Stability properties for quasilinear parabolic equations with measure data and applications

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Abstract

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$. We first study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu \qquad \text{in } Q, \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \qquad \text{in } \Omega, \end{cases}$$

where $p > 1$, $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators $u \longmapsto \mathcal{A}(u) = \mathrm{div}(A(x,t,\nabla u))$.

As an application, we consider perturbed problems *of type*

$$\begin{cases} u_t - \Delta_p u + \mathcal{G}(u) = \mu \qquad \text{in } Q, \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \qquad \text{in } \Omega, \end{cases}$$

where $\mathcal{G}(u)$ may be an absorption or a source term. In the model case $\mathcal{G}(u) = \pm |u|^{q-1} u$ ($q > p-1$), or $\mathcal{G}$ has an exponential type. We give existence results when q is subcritical, or when the measure μ is good in time and satisfies suitable capacity conditions.

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively. We are concerned with the problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and A is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for *a.e.* $(x,t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,t,\xi).\xi \geqq c_1 |\xi|^p, \qquad |A(x,t,\xi)| \leqq a(x,t) + c_2 |\xi|^{p-1}, \qquad c_1, c_2 > 0, a \in L^{p'}(Q), \tag{1.2}$$

$$(A(x,t,\xi) - A(x,t,\zeta)).(\xi - \zeta) > 0 \qquad \text{if } \xi \neq \zeta. \tag{1.3}$$

This includes the model problem

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

where Δ_p is the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$.
As an application, we consider problems with a nonlinear term of order 0:

$$\begin{cases} u_t - \operatorname{div}(A(x,\nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where A is a Caratheodory function on $\Omega \times \mathbb{R}^N$, such that, for *a.e.* $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,\xi).\xi \geqq c_1 |\xi|^p, \qquad |A(x,\xi)| \leqq c_2 |\xi|^{p-1}, \qquad c_3, c_4 > 0, \tag{1.6}$$

$$(A(x,\xi) - A(x,\zeta)).(\xi - \zeta) > 0 \text{ if } \xi \neq \zeta, \tag{1.7}$$

and $\mathcal{G}(u)$ may be an absorption or a source term, and possibly depends on $(x,t) \in Q$. The model problem is the case where $\mathcal{G}$ has a power-type $\mathcal{G}(u) = \pm |u|^{q-1} u$ ($q > p-1$), or an exponential type.

First make a brief survey of the elliptic associated problem:

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\mu \in \mathcal{M}_b(\Omega)$ and assumptions (1.6), (1.7). When $p = 2$, $A(x,\nabla u) = \nabla u$ existence and uniqueness are proved for general elliptic operators by duality methods in [58]. For $p > 2 - 1/N$, the existence of solutions in the sense of distributions is obtained in [23] and [24]. The condition on p ensures that the gradient ∇u is well defined in $(L^1(\Omega))^N$. For general $p > 1$, new classes of solutions are introduced, first when $\mu \in L^1(\Omega)$, such as *entropy solutions*, and *renormalized solutions*, see [13], and also [57], and existence and uniqueness is obtained. For any $\mu \in \mathcal{M}_b(\Omega)$ the main work is done in [32, Theorems 3.1, 3.2], where not only existence is proved, but also a stability result, fundamental for applications. Uniqueness is still an open problem.

Next we make a brief survey about problem (1.1).

The first studiess concern the case $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness is obtained by variational methods, see [45]. In the general case $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$, the pionner results come from [23], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \tag{1.8}$$

see also [55], [56], and [26]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces $u \in L^{p_c,\infty}(Q)$ and $|\nabla u| \in L^{m_c,\infty}(Q)$, where

$$p_c = p - 1 + \frac{p}{N}, \qquad m_c = p - \frac{N}{N+1}. \tag{1.9}$$

This condition (1.8) ensures that u and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N+1)$. Uniqueness follows in the case $p = 2$, $A(x,t,\nabla u) = \nabla u$, by duality methods, see [48].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of solutions: *entropy solutions*, and *renormalized solutions*, see [19], [54], see also [3] for a semi-group approach.

Then a class of *regular* measures is studied in [33], where a notion of parabolic capacity c_p^Q is introduced, defined by

$$c_p^Q(E) = \inf\left(\inf_{E\subset U \text{ open}\subset Q}\{||u||_W : u \in W, u \geqq \chi_U \quad a.e. \text{ in } Q\}\right),$$

for any Borel set $E \subset Q$, where

$$X = L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega)),$$
$$W = \left\{z : z \in X, \quad z_t \in X'\right\}, \text{ embedded with the norm } ||u||_W = ||u||_X + ||u_t||_{X'}.$$

Let $\mathcal{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c_p^Q-capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \Longrightarrow |\mu(E)| = 0.$$

Then existence and uniqueness of renormalized solutions holds for any measure $\mu \in \mathcal{M}_b(\Omega) \cap \mathcal{M}_0(Q)$, called regular (or diffuse) and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [34]; see also [20] for more general equations.

Next consider *any* measure $\mu \in \mathcal{M}_b(Q)$. Let $\mathcal{M}_s(Q)$ be the set of all bounded Radon measures on Q with support on a set of zero c_p^Qcapacity, also called *singular*. Let $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$

be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$. From [33], μ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \qquad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \qquad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \tag{1.10}$$

and $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \qquad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \tag{1.11}$$

and we write $\mu_0 = (f, g, h)$. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [19],[49]. In the range (1.8) the existence of a renormalized solution relative to the decomposition (1.11) is proved in [49], using suitable approximations of μ_0 and μ_s. Uniqueness is still open, as well as in the elliptic case.

Next consider the problem (1.5). First we consider the case of an *absorption term*: $\mathcal{G}(u)u \geqq 0$. Let us recall the case $p = 2$, $A(x, \nabla u) = \nabla u$ and $\mathcal{G}(u) = |u|^{q-1}u$ $(q > 1)$. The first results concern the case $\mu = 0$ and u_0 is a Dirac mass in Ω, see [28]: existence holds if and only if $q < (N+2)/N$. Then optimal results are given in [7], for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $C_{\alpha,k}$, $(\alpha, k > 1)$ defined, for any Borel set $E \subset \mathbb{R}^N$, by

$$C_{\alpha,k}(E) = \inf\{||\varphi||_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N), \quad G_\alpha * \varphi \geqq \chi_E\},$$

where G_α is the Bessel kernel of order α; and a capacity $c_{\mathbf{G},k}$ $(k > 1)$ adapted to the operator of the heat equation of kernel $\mathbf{G}(x,t) = \chi_{(0,\infty)}(4\pi t)^{-N/2}e^{-|x|^2/4t}$: for any Borel set $E \subset \mathbb{R}^{N+1}$,

$$c_{\mathbf{G},k}(E) = \inf\{||\varphi||_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}), \quad \mathbf{G} * \varphi \geqq \chi_E\}.$$

From [7], there exists a solution if and only if μ does not charge the sets of $c_{\mathbf{G},q'}(E)$ capacity zero and u_0 does not charge the sets of $C_{2/q,q'}$ capacity zero. Observe that one can reduce to a zero initial data, by considering the measure $\mu + u_0 \otimes \delta_0^t$ in $\Omega \times (-T, T)$, where $\otimes$ is the tensorial product and δ_0^t is the Dirac mass in time at 0.

For $p \neq 2$ such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case $\mu = 0$ with Ω bounded, or $\Omega = \mathbb{R}^N$. The case where u_0 is a Dirac mass in Ω is studied in [36], [40] when $p > 2$, and [29] when $p < 2$. Existence and uniqueness hold in the subcritical case $q < p_c$. If $q \geqq p_c$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathcal{M}_b^+(\Omega)$, the existence is obtained in the sense of distributions in [60], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [16]. The case $\mu \in L^1(Q)$, $u_0 = 0$ is treated in [30], and $\mu \in L^1(Q)$, $u_0 = L^1(\Omega)$ in [4] where $\mathcal{G}$ can be multivalued. The case $\mu \in \mathcal{M}_0(Q)$ is studied in [50], with a new formulation of the solutions, and existence and uniqueness is obtained for any function $\mathcal{G} \in C(\mathbb{R})$ such that $\mathcal{G}(u)u \geqq 0$. Up to our knowledge, up to now no existence results have been obtained for a general measure $\mu \in \mathcal{M}_b(Q)$.

The case of a source term $\mathcal{G}(u) = -u^q$ with $u \geqq 0$ has beeen treated in [6] for $p = 2$, where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general p.

2 Main results

In *all the sequel* we suppose that p satisfies (1.8). Then

$$X = L^p(0,T;W_0^{1,p}(\Omega)), \qquad X' = L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

We first study problem (1.1). In Section 3 we give some approximations of $\mu \in \mathcal{M}_b(Q)$, useful for the applications. In Section 4 we recall the definition of renormalized solutions, that we call R-solutions of (1.1), relative to the decomposition (1.11) of μ_0, and study some of their properties.

Our main result is a *stability theorem* for problem (1.1), proved in Section 5, extending to the parabolic case the stability result of [32, Theorem 3.4], and improving the result of [49]:

Theorem 2.1 *Let $A : Q \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ satisfying (1.2),(1.3). Let $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

with $f \in L^1(Q), g \in (L^{p'}(Q))^N$, $h \in X$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

with $f_n \in L^1(Q), g_n \in (L^{p'}(Q))^N, h_n \in X$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q), \rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n|(Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in X, $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology of measures; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$. Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} u_{n,t} - div(A(x,t,\nabla u_n)) = \mu_n & in\ Q, \\ u_n = 0 & on\ \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} & in\ \Omega. \end{cases} \tag{2.1}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $v_n = u_n - h_n$. Then up to a subsequence, $\{u_n\}$ converges a.e. in Q to a R-solution u of (1.1), and $\{v_n\}$ converges a.e. in Q to $v = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla v_n\}$ converge respectively to $\nabla u, \nabla v$ a.e. in Q, and $\{T_k(u_n)\}, \{T_k(v_n)\}$ converge to $T_k(u)$, $T_k(v)$ strongly in X for any $k > 0$.

In Section 6 we give applications to problems of type (1.5).

We first give an existence result of subcritical type, valid for any measure $\mu \in \mathcal{M}_b(Q)$:

Theorem 2.2 *Let $A : Q \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.2), (1.3) with $a \equiv 0$. Let $(x,t,r) \mapsto \mathcal{G}(x,t,r)$ be a Caratheodory function on $Q \times \mathbb{R}$ and $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in $\mathbb{R}^+$, such that*

$$|\mathcal{G}(x,t,r)| \leqq G(|r|) \quad \text{for a.e. } (x,t) \in Q \text{ and any } r \in \mathbb{R}, \tag{2.2}$$

$$\int_1^\infty G(s)s^{-1-p_c}ds < \infty. \tag{2.3}$$

(i) Suppose that $\mathcal{G}(x,t,r)r \geqq 0$, for a.e. (x,t) in Q and any $r \in \mathbb{R}$. Then, for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a R-solution u of problem

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{2.4}$$

(ii) Suppose that $\mathcal{G}(x,t,r)r \leqq 0$, for a.e. $(x,t) \in Q$ and any $r \in \mathbb{R}$, and $u_0 \geqq 0, \mu \geqq 0$. There exists $\varepsilon > 0$ such that for any $\lambda > 0$, any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$ with $\lambda + |\mu|(Q) + ||u_0||_{L^1(\Omega)} \leqq \varepsilon$, problem

$$\begin{cases} u_t - div(A(x,t,\nabla u)) + \lambda\mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{2.5}$$

admits a nonnegative R-solution.

In particular for any $0 < q < p_c$, if $\mathcal{G}(u) = |u|^{q-1} u$, existence holds for any measure $\mu \in \mathcal{M}_b(Q)$; if $\mathcal{G}(u) = -|u|^{q-1} u$, existence holds for μ small enough. In the supercritical case $q \geqq p_c$, the class of "admissible" measures, for which there exist solutions, is not known.

Next we give new results relative to *measures that have a good behaviour in t*, based on recent results of [17] relative to the elliptic case. We recall the notions of (truncated) Wölf potential for any nonnegative measure $\omega \in \mathcal{M}^+(\mathbb{R}^N)$ any $R > 0$, $x_0 \in \mathbb{R}^N$,

$$\mathbf{W}_{1,p}^R[\omega](x_0) = \int_0^R \left(t^{p-N}\omega(B(x_0,t))\right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Any measure $\omega \in \mathcal{M}_b(\Omega)$ is identified with its extension by 0 to $\mathbb{R}^N$. In case of absorption, we obtain the following:

Theorem 2.3 *Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.6),(1.7). Let $p < N$, $q > p-1$, $\mu \in \mathcal{M}_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that*

$$|\mu| \leqq \omega \otimes F, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega), F \in L^1((0,T)), F \geqq 0,$$

and ω does not charge the sets of $C_{p,\frac{q}{q+1-p}}$-capacity zero. Then there exists a R- solution u of problem

$$\begin{cases} u_t - div(A(x,\nabla u)) + |u|^{q-1}u = f + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{2.6}$$

We show that some of these measures may not lie in $\mathcal{M}_0(Q)$, which improves the existence results of [50], see Proposition 3.3 and Remark 6.7. Otherwise our result can be extended to a more general function $\mathcal{G}$, see Remark 6.9. We also consider a source term:

Theorem 2.4 *Let $A : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N$, $q > p-1$. Let $\mu \in \mathcal{M}_b^+(Q)$, and $u_0 \in L^\infty(\Omega), u_0 \geqq 0$. Assume that*

$$\mu \leqq \omega \otimes \chi_{(0,T)}, \qquad \text{with } \omega \in \mathcal{M}_b^+(\Omega).$$

Then there exist $\lambda_0 = \lambda_0(N, p, q, c_3, c_4, diam\Omega)$ and $b_0 = b_0(N, p, q, c_3, c_4, \mathrm{diam}\Omega)$ such that, if

$$\omega(E) \leqq \lambda_0 C_{p,\frac{q}{q-p+1}}(E), \quad \forall E \text{ compact} \subset \mathbb{R}^N, \qquad ||u_0||_{\infty,\Omega} \leqq b_0, \tag{2.7}$$

there exists a nonnegative R-solution u of problem

$$\begin{cases} u_t - div(A(x, \nabla u)) = u^q + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{2.8}$$

which satisfies, a.e. in Q,

$$u(x, t) \leqq C\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega](x) + 2||u_0||_{L^\infty}, \tag{2.9}$$

where $C = C(N, p, c_3, c_4)$.

Corresponding results in case where $\mathcal{G}$ has exponential type are given at Theorems 6.10 and 6.15.

3 Approximations of measures

For any open set ϖ of $\mathbb{R}^m$ and $F \in (L^k(\varpi))^\nu$, $k \in [1,\infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k,\varpi} = \|F\|_{(L^k(\varpi))^\nu}$.

First we give approximations of nonnegative measures in $\mathcal{M}_0(Q)$. We recall that any measure $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$ admits a decomposition under the form $\mu = (f, g, h)$ given by (1.11). Conversely, any measure of this form, *such that* $h \in L^\infty(Q)$, lies in $\mathcal{M}_0(Q)$, see [50, Proposition 3.1].

Lemma 3.1 *Let $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$.*
(i) Then, we can find a decomposition $\mu = (f, g, h)$ with $f \in L^1(Q), g \in (L^{p'}(Q))^N, h \in X$ such that

$$||f||_{1,Q} + \|g\|_{p',Q} + ||h||_X \leqq (1 + \varepsilon)\mu(Q), \qquad \|g\|_{p',Q} + ||h||_X \leqq \varepsilon. \tag{3.1}$$

(ii) Furthermore, there exists a sequence of measures $\mu_n = (f_n, g_n, h_n)$, such that $f_n, g_n, h_n \in C_c^\infty(Q)$ and strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and μ_n converges to μ in the narrow topology, and satisfying

$$||f_n||_{1,Q} + \|g_n\|_{p',Q} + ||h_n||_X \leqq (1 + 2\varepsilon)\mu(Q), \qquad \|g_n\|_{p',Q} + ||h_n||_X \leqq 2\varepsilon. \tag{3.2}$$

Proof. (i) Step 1. Case where μ has a compact support in Q. By [33], we can find a decomposition $\mu = (f, g, h)$ with f, g, h have a compact support in Q. Let $\{\varphi_n\}$ be sequence of mollifiers in $\mathbb{R}^{N+1}$. Then $\mu_n = \varphi_n * \mu \in C_c^\infty(Q)$ for n large enough. We see that $\mu_n(Q) = \mu(Q)$ and μ_n admits the decomposition $\mu_n = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$. Since $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, we have for n_0 large enough,

$$||f - f_{n_0}||_{1,Q} + ||g - g_{n_0}||_{p',Q} + ||h - h_{n_0}||_X \leqq \varepsilon \min\{\mu(Q), 1\}.$$

Then we obtain a decomposition $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$, such that

$$||\hat{f}||_{1,Q} + ||\hat{g}||_{p',Q} + ||\hat{h}||_X \leqq (1+\varepsilon)\mu(Q), \qquad \|\hat{g}\|_{p',Q} + ||\hat{h}||_X \leqq \varepsilon. \tag{3.3}$$

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Set $\tilde{\mu}_0 = \theta_0\mu$, and $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$, for any $n \geqq 1$. Since $\tilde{\mu}_n \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ has compact support, by Step 1, we can find a decomposition $\tilde{\mu}_n = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ such that

$$||\tilde{f}_n||_{1,Q} + \|\tilde{g}_n\|_{p',Q} + ||\tilde{h}_n||_X \leqq (1+\varepsilon)\mu_n(Q), \qquad ||\tilde{g}_n||_{p',Q} + ||\tilde{h}_n||_X \leqq 2^{-n-1}\varepsilon.$$

Let $\overline{f}_n = \sum_{k=0}^{n} \tilde{f}_k$, $\overline{g}_n = \sum_{k=0}^{n} \tilde{g}_k$ and $\bar{h}_n = \sum_{k=0}^{n} \tilde{h}_k$. Clearly, $\theta_n\mu = (\overline{f}_n, \overline{g}_n, \bar{h}_n)$, and $\{\overline{f}_n\}$,$\{\overline{g}_n\}$, $\{\bar{h}_n\}$ converge strongly to some f, g, h, respectively in $L^1(Q)$, $(L^{p'}(Q))^N$, X, with

$$||\overline{f}_n||_{1,Q} + ||\overline{g}_n||_{p',Q} + ||\bar{h}_n||_X \leqq (1+\varepsilon)\mu(Q), \qquad ||\overline{g}_n||_{p',Q} + ||\bar{h}_n||_X \leqq \varepsilon.$$

Therefore, $\mu = (f, g, h)$ and (3.1) holds.

(ii) We take a sequence $\{m_n\}$ in $\mathbb{N}$ such that $f_n = \varphi_{m_n} * \overline{f}_n, g_n = \varphi_{m_n} * \overline{g}_n, h_n = \varphi_{m_n} * \bar{h}_n \in C_c^\infty(Q)$ and

$$||f_n - \overline{f}_n||_{1,Q} + ||g_n - \overline{g}_n||_{p',Q} + ||h_n - \bar{h}_n||_X \leqq \frac{\varepsilon}{n+1}\min\{\mu(Q), 1\}.$$

Let $\mu_n = \varphi_{m_n} * (\theta_n\mu) = (f_n, g_n, h_n)$. Therefore, $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively. And (3.2) holds. Furthermore, $\{\mu_n\}$ converges weak-* to μ, and $\mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu(Q)$, thus $\{\mu_n\}$ converges in the narrow topology. ■

As a consequence, we get an approximation property for any measure $\mu \in \mathcal{M}_b^+(Q)$:

Proposition 3.2 *Let $\mu \in \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to μ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q)$, $g_n, g \in (L^{p'}(Q))^N$ and $h_n, h \in X$, $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \qquad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and $\{\mu_{n,s}\}$ converges to μ_s (strongly) in $\mathcal{M}_b(Q)$ and

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X + \mu_{n,s}(\Omega) \leqq (1+\varepsilon)\mu(Q), \qquad \textit{and } ||g_n||_{p',Q} + ||h_n||_X \leqq \varepsilon. \tag{3.4}$$

Proof. Since $\{\mu_n\}$ is nondecreasing, then $\{\mu_{n,0}\}$, $\{\mu_{n,s}\}$ are too. Clearly, $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$. Hence, $\{\mu_{n,s}\}$ converge to μ_s and $\{\mu_{n,0}\}$ converge to μ_0 (strongly) in $\mathcal{M}_b(Q)$. Set $\widetilde{\mu}_{0,0} = \mu_{0,0}$, and $\widetilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for any $n \geqq 1$. By Lemma 3.1, (i), we can find $\tilde{f}_n \in L^1(Q)$, $\tilde{g}_n \in (L^{p'}(Q))^N$ and $\tilde{h}_n \in X$ such that $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ and

$$||\tilde{f}_n||_{1,Q} + ||\tilde{g}_n||_{p',Q} + ||\tilde{h}_n||_X \leqq (1+\varepsilon)\tilde{\mu}_{n,0}(Q), \qquad ||\tilde{g}_n||_{p',Q} + ||\tilde{h}_n||_X \leqq 2^{-n-1}\varepsilon.$$

Let $f_n = \sum\limits_{k=0}^{n} \tilde{f}_k$, $G_n = \sum\limits_{k=0}^{n} \tilde{g}_k$ and $h_n = \sum\limits_{k=0}^{n} \tilde{h}_k$. Clearly, $\mu_{n,0} = (f_n, g_n, h_n)$ and the convergence properties hold with (3.4), since

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X \leqq (1+\varepsilon)\mu_0(Q).$$

■

In Section 6 we consider some measures $\mu \in \mathcal{M}_b(Q)$ which satisfy $|\mu| \leqq \omega \otimes F$, with $\omega \in \mathcal{M}_b(\Omega)$ and $F \in L^1((0,T)), F \geqq 0$. It is interesting to compare the properties of $\omega \otimes F$ and ω :

Let c_p^Ω be the elliptic capacity in Ω defined by

$$c_p^\Omega(K) = \inf\{\int_\Omega |\nabla\varphi|^p : \varphi \geqq \chi_K, \varphi \in C_c^\infty(\Omega)\},$$

for any compact set $K \subset \Omega$.

Let $\mathcal{M}_{0,e}(\Omega)$ be the set of Radon measures ω on that do not charge the sets of zero c_p^Ω-capacity. Then $\mathcal{M}_b(\Omega) \cap \mathcal{M}_{0,e}(\Omega)$ is characterised as the set of measures $\omega \in \mathcal{M}_b(\Omega)$ which can be written under the form $\tilde{f} - \operatorname{div}\tilde{g}$ with $\tilde{f} \in L^1(\Omega)$ and $\tilde{g} \in (L^{p'}(\Omega))^N$, see [25].

Proposition 3.3 *For any $F \in L^1((0,T))$ with $\int_0^T F(t)dt \neq 0$, and $\omega \in \mathcal{M}_b(\Omega)$,*

$$\omega \in \mathcal{M}_{0,e}(\Omega) \Longleftrightarrow \omega \otimes F \in \mathcal{M}_0(Q).$$

Proof. Assume that $\omega \otimes F \in \mathcal{M}_0(Q)$. Then, there exist $f \in L^1(Q)$, $g \in \left(L^{p'}(Q)\right)^N$ and $h \in X$, such that

$$\int_Q \varphi(x,t)F(t)d\omega(x)dt = \int_Q \varphi(x,t)f(x,t)dxdt + \int_Q g(x,t).\nabla\varphi(x,t)dxdt - \int_Q h(x,t)\varphi_t(t,x)dxdt, \tag{3.5}$$

for all $\varphi \in C_c^\infty(\Omega \times [0,T])$, see [50, Lemma 2.24 and Theorem 2.27]. By choosing $\varphi(x,t) = \varphi(x) \in C_c^\infty(\Omega)$ and using Fubini's Theorem, (3.5) is rewritten as

$$\int_\Omega \varphi(x)d\omega(x) = \int_\Omega \varphi(x)\tilde{f}(x)dx + \int_\Omega \tilde{g}(x).\nabla\varphi(x)dx,$$

where $\tilde{f}(x) = \left(\int_0^T F(t)dt\right)^{-1} \int_0^T f(x,t)dt \in L^1(\Omega)$ and $\tilde{g}(x) = \left(\int_0^T F(t)dt\right)^{-1} \int_0^T g(x,t)dt \in \left(L^{p'}(\Omega)\right)^N$; hence $\omega \in \mathcal{M}_{0,e}(\Omega)$.

Conversely, assume that $\omega = \tilde{f} - \operatorname{div} \tilde{g} \in \mathcal{M}_{0,e}(\Omega)$, with $\tilde{f} \in L^1(\Omega)$ and $\tilde{g} \in \left(L^{p'}(\Omega)\right)^N$. So $\omega \otimes T_n(F) = f_n - \operatorname{div} g_n$, with $f_n = \tilde{f} T_n(F) \in L^1(Q)$ and $g_n = \tilde{g} T_n(F) \in \left(L^{p'}(Q)\right)^N$. Then $\omega \otimes T_n(F)$ admits the decomposition (f_n, g_n, h), with $h = 0 \in L^\infty(Q)$, thus $\omega \otimes T_n(F) \in \mathcal{M}_0(Q)$. And $\{\omega \otimes T_n(F)\}$ converges to $\omega \otimes F$ strongly in $\mathcal{M}_b(Q)$, since $||\omega \otimes (F - T_n(F))||_{\mathcal{M}_b(Q)} \leqq ||\omega||_{\mathcal{M}_b(\Omega)} \|F - T_n(F)\|_{L^1((0,T))}$. Then $\omega \otimes F \in \mathcal{M}_0(Q)$, since $\mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$ is strongly closed in $\mathcal{M}_b(Q)$. ■

4 Renormalized solutions of problem (1.1)

4.1 Notations and Definition

For any function $f \in L^1(Q)$, we write $\int_Q f$ instead of $\int_Q f dxdt$, and for any measurable set $E \subset Q$, $\int_E f$ instead of $\int_E f dxdt$.

We set $T_k(r) = \max\{\min\{r, k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. We recall that if u is a measurable function defined and finite *a.e.* in Q, such that $T_k(u) \in X$ for any $k > 0$, there exists a measurable function w from Q into $\mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \leqq k} w$, *a.e.* in Q, and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, and (f, g, h) be a decomposition of μ_0 given by (1.11), and $\widehat{\mu_0} = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu_0} \notin \mathcal{M}(Q)$, but we write, for convenience,

$$\int_Q w d\widehat{\mu_0} := \int_Q (fw + g.\nabla w), \qquad \forall w \in X \cap L^\infty(Q).$$

Definition 4.1 *Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function u is a* ***renormalized solution,*** *called* ***R-solution*** *of (1.1) if there exists a decompostion (f, g, h) of μ_0 such that*

$$v = u - h \in L^\sigma(0, T; W_0^{1,\sigma}(\Omega) \cap L^\infty(0, T; L^1(\Omega)), \quad \forall \sigma \in [1, m_c)\,; \qquad T_k(v) \in X, \quad \forall k > 0, \tag{4.1}$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$-\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x,t,\nabla u).\nabla\varphi + \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v = \int_Q S'(v)\varphi d\widehat{\mu_0}, \tag{4.2}$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(T, .) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \leqq v < 2m\}} \phi A(x,t,\nabla u).\nabla v = \int_Q \phi d\mu_s^+ \tag{4.3}$$

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{-m\geqq v>-2m\}}\phi A(x,t,\nabla u).\nabla v=\int_Q \phi d\mu_s^-. \tag{4.4}$$

Remark 4.2 *As a consequence, $S(v)\in C([0,T];L^1(\Omega))$ and $S(v)(0,.)=S(u_0)$ in Ω; and u satisfies the equation*

$$(S(v))_t-\operatorname{div}(S'(v)A(x,t,\nabla u))+S''(v)A(x,t,\nabla u).\nabla v= fS'(v)-\operatorname{div}(gS'(v))+S''(v)g.\nabla v, \tag{4.5}$$

in the sense of distributions in Q, see [49, Remark 3]. Moreover

$$\begin{aligned}\|S(v)_t\|_{X'+L^1(Q)} \leqq{} & \left\|\operatorname{div}(S'(v)A(x,t,\nabla u))\right\|_{X'}+\left\|S''(v)A(x,t,\nabla u).\nabla v\right\|_{1,Q}\\ &+\left\|S'(v)f\right\|_{1,Q}+\left\|g.S''(v)\nabla v\right\|_{1,Q}+\left\|\operatorname{div}(S'(v)g)\right\|_{X'}.\end{aligned}$$

Thus, if $[-M,M]\supset suppS'$,

$$\begin{aligned}&\left\|S''(v)A(x,t,\nabla u).\nabla v\right\|_{1,Q}\leq \|S\|_{W^{2,\infty}(\mathbb{R})}\left(\left\|A(x,t,\nabla u)\chi_{|v|\leqq M}\right\|_{p',Q}^{p'}+\||\nabla T_M(v)|\|_{p,Q}^p\right)\\ &\leqq C\,\|S\|_{W^{2,\infty}(\mathbb{R})}\left(\left\||\nabla u|^p\chi_{|v|\leqq M}\right\|_{1,Q}+\|a\|_{p',Q}^{p'}+\||\nabla T_M(v)|\|_{p,Q}^p\right)\end{aligned}$$

thus

$$\begin{aligned}\|S(v)_t\|_{X'+L^1(Q)}\leqq{}& C\,\|S\|_{W^{2,\infty}(\mathbb{R})}\Big(\left\||\nabla u|^p\chi_{|v|\leqq M}\right\|_{1,Q}^{1/p'}+\left\||\nabla u|^p\chi_{|v|\leqq M}\right\|_{1,Q}+\||\nabla T_M(v)|\|_{p,Q}^p\\ &+\|a\|_{p',Q}+\|a\|_{p',Q}^{p'}+\|f\|_{1,Q}+\|g\|_{p',Q}\left\||\nabla u|^p\chi_{|v|\leqq M}\right\|_{1,Q}^{1/p}+\|g\|_{p',Q}\Big)\end{aligned} \tag{4.6}$$

We also deduce that, for any $\varphi\in X\cap L^\infty(Q)$, such that $\varphi_t\in X'+L^1(Q)$,

$$\begin{aligned}&\int_\Omega S(v(T))\varphi(T)dx-\int_\Omega S(u_0)\varphi(0)dx-\int_Q\varphi_tS(v)+\int_Q S'(v)A(x,t,\nabla u).\nabla\varphi\\ &\qquad+\int_Q S''(v)A(x,t,\nabla u).\nabla v\varphi=\int_Q S'(v)\varphi d\widehat{\mu_0}.\end{aligned} \tag{4.7}$$

Remark 4.3 *Let u,v satisfying (4.1). It is easy to see that the condition (4.3) (resp. (4.4)) is equivalent to*

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\leqq v<2m\}}\phi A(x,t,\nabla u).\nabla u=\int_Q\phi d\mu_s^+ \tag{4.8}$$

resp.

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\geqq v>-2m\}}\phi A(x,t,\nabla u).\nabla u=\int_Q\phi d\mu_s^-. \tag{4.9}$$

In particular, for any $\varphi\in L^{p'}(Q)$ there holds

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{m\leqq|v|<2m}|\nabla u|\varphi=0,\qquad \lim_{m\to\infty}\frac{1}{m}\int\limits_{m\leqq|v|<2m}|\nabla v|\varphi=0. \tag{4.10}$$

Remark 4.4 *(i) Any function $U \in X$ such that $U_t \in X' + L^1(Q)$ admits a unique c_p^Q-quasi continuous representative, defined c_p^Q-quasi a.e. in Q, still denoted U. Furthermore, if $U \in L^\infty(Q)$, then for any $\mu_0 \in \mathcal{M}_0(Q)$, there holds $U \in L^\infty(Q, d\mu_0)$, see [49, Theorem 3 and Corollary 1].*

(ii) Let u be any R- solution of problem (1.1). Then, $v = u - h$ admits a c_p^Q-quasi continuous functions representative which is finite c_p^Q-quasi a.e. in Q, and u satisfies definition 4.1 for every decomposition $(\tilde{f}, \tilde{g}, \tilde{h})$ such that $h - \tilde{h} \in L^\infty(Q)$, see [49, Proposition 3 and Theorem 4].

4.2 Steklov and Landes approximations

A main difficulty for proving Theorem 2.1 is the choice of admissible test functions (S, φ) in (4.2), valid for any R-solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

Definition 4.5 *Let $\varepsilon \in (0,T)$ and $z \in L^1_{loc}(Q)$. For any $l \in (0, \varepsilon)$ we define the* ***Steklov time-averages*** *$[z]_l, [z]_{-l}$ of z by*

$$[z]_l(x,t) = \frac{1}{l}\int_t^{t+l} z(x,s)ds \qquad \text{for a.e. } (x,t) \in \Omega \times (0, T-\varepsilon),$$

$$[z]_{-l}(x,t) = \frac{1}{l}\int_{t-l}^{t} z(x,s)ds \qquad \text{for a.e. } (x,t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for R-solutions can be found in [22]. Recall some properties, given in [50]. Let $\varepsilon \in (0,T)$, and $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0,T))$, $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0,T])$ with $\mathrm{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T-\varepsilon]$, $\mathrm{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$. There holds

(i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.

(ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \to 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in X, and *a.e.* in Q; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.

(iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \to 0$, there exists a subsequence $\{l_\nu\}$ such that $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$ converge to z, c_p^Q-quasi everywhere in Q.

Next we recall the approximation introduced in [42], used in [30], [26], [21]:

Definition 4.6 *Let $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Let u be a R-solution of (1.1), and $v = u - h$ given at (4.1), and $k > 0$. For any $\nu \in \mathbb{N}$, the* ***Landes-time approximation*** *$\langle T_k(v)\rangle_\nu$ of the truncate function $T_k(v)$ is defined in the following way:*

Let $\{z_\nu\}$ be a sequence of functions in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, such that $||z_\nu||_{\infty,\Omega} \leqq k$, $\{z_\nu\}$ converges to $T_k(u_0)$ a.e. in Ω, and $\nu^{-1}||z_\nu||^p_{W_0^{1,p}(\Omega)}$ converges to 0. Then, $\langle T_k(v)\rangle_\nu$ is the unique solution of the problem

$$(\langle T_k(v)\rangle_\nu)_t = \nu\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) \quad \textit{in the sense of distributions,} \quad \langle T_k(v)\rangle_\nu(0) = z_\nu, \qquad \textit{in } \Omega.$$

Therefore, $\langle T_k(v)\rangle_\nu \in X \cap L^\infty(Q)$ and $(T_k(v)\rangle_\nu)_t \in X$, see [42]. Furthermore, up to subsequences, $\{\langle T_k(v)\rangle_\nu\}$ converges to $T_k(v)$ strongly in X and *a.e.* in Q, and $||\left(T_k(v)\right)_\nu||_{L^\infty(Q)} \leqq k$.

4.3 First properties

In the sequel we use the following notations: for any function $J \in W^{1,\infty}(\mathbb{R})$, nondecreasing with $J(0) = 0$, we set

$$\overline{J}(r) = \int_0^r J(\tau)d\tau, \qquad \mathcal{J}(r) = \int_0^r J'(\tau)\tau d\tau. \tag{4.11}$$

It is easy to verify that $\mathcal{J}(r) \geqq 0$,

$$\mathcal{J}(r) + \overline{J}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geqq s\left(J(r) - J(s)\right) \qquad \forall r, s \in \mathbb{R}. \tag{4.12}$$

In particular we define, for any $k > 0$, and any $r \in \mathbb{R}$,

$$\overline{T_k}(r) = \int_0^r T_k(\tau)d\tau, \qquad \mathcal{T}_k(r) = \int_0^r T_k'(\tau)\tau d\tau, \tag{4.13}$$

and we use several times a truncature used in [32]:

$$H_m(r) = \chi_{[-m,m]}(r) + \frac{2m - |s|}{m}\chi_{m<|s|\leqq 2m}(r), \qquad \overline{H_m}(r) = \int_0^r H_m(\tau)d\tau. \tag{4.14}$$

The next Lemma allows to extend the range of the test functions in (4.2). Its proof, given in the Appendix, is obtained by Steklov approximation of the solutions.

Lemma 4.7 *Let u be a R-solution of problem (1.1). Let $J \in W^{1,\infty}(\mathbb{R})$ be nondecreasing with $J(0) = 0$, and $\overline{J}$ defined by (4.11). Then,*

$$\begin{aligned}
&\int_Q S'(v)A(x,t,\nabla u).\nabla\left(\xi J(S(v))\right) + \int_Q S''(v)A(x,t,\nabla u).\nabla v \xi J(S(v)) \\
&- \int_\Omega \xi(0)J(S(u_0))S(u_0) - \int_Q \xi_t \overline{J}(S(v)) \\
&\leqq \int_Q S'(v)\xi J(S(v))d\widehat{\mu_0},
\end{aligned} \tag{4.15}$$

for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0) = 0$, and for any $\xi \in C^1(Q) \cap W^{1,\infty}(Q), \xi \geqq 0$.

Next we give estimates of the gradient, following the first estimates of [26], see also [33], [49, Proposition 2], [43].

Proposition 4.8 *If u is a R-solution of problem (1.1), then there exists $c = c(p)$ such that, for any $k \geqq 1$ and $\ell \geqq 0$,*

$$\int_{\ell \leqq |v| \leqq \ell+k} |\nabla u|^p + \int_{\ell \leqq |v| \leqq \ell+k} |\nabla v|^p \leqq ckM \tag{4.16}$$

and

$$\|v\|_{L^\infty((0,T);L^1(\Omega))} \leqq c(M + |\Omega|), \tag{4.17}$$

where

$$M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + ||a||_{p',Q}^{p'}.$$

As a consequence, for any $k \geqq 1$,

$$\text{meas}\{|v| > k\} \leqq C_1 M_1 k^{-p_c}, \qquad \text{meas}\{|\nabla v| > k\} \leqq C_2 M_2 k^{-m_c}, \tag{4.18}$$

$$\text{meas}\{|u| > k\} \leqq C_3 M_2 k^{-p_c}, \qquad \text{meas}\{|\nabla u| > k\} \leqq C_4 M_2 k^{-m_c}, \tag{4.19}$$

where $C_i = C_i(N, p, c_1, c_2)$, $i = 1$-4, and $M_1 = (M+|\Omega|)^{\frac{p}{N}} M$ and $M_2 = M_1 + M$.

Proof. Set for any $r \in \mathbb{R}$, and $m, k, \ell > 0$,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For $m > k+\ell$, we can choose $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$ as test functions in (4.15), where $\overline{H_m}$ is defined at (4.14) and $\xi \in C^1([0,T])$ with values in $[0,1]$, independent on x. Since $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$ for all $r \in \mathbb{R}$, we obtain

$$\begin{aligned}
&-\textstyle\int_\Omega \xi(0) T_{k,\ell}(u_0)\overline{H_m}(u_0) - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(v)) \\
&+ \int_{\{\ell \leqq |v| < \ell+k\}} \xi A(x,t,\nabla u).\nabla v - \frac{k}{m} \int_{\{m \leqq |v| < 2m\}} \xi A(x,t,\nabla u).\nabla v \leqq \textstyle\int_Q H_m(v)\xi T_{k,\ell}(v) d\widehat{\mu_0}.
\end{aligned}$$

And

$$\int_Q H_m(v)\xi T_{k,\ell}(v) d\widehat{\mu_0} = \int_Q H_m(v)\xi T_{k,\ell}(v) f + \int_{\{\ell \leqq |v| < \ell+k\}} \xi \nabla v.g - \frac{k}{m} \int_{\{m \leqq |v| < 2m\}} \xi \nabla v.g.$$

Let $m \to \infty$; then, for any $k \geqq 1$, since $v \in L^1(Q)$ and from (4.3), (4.4), and (4.10), we find

$$-\int_Q \xi_t \overline{T_{k,\ell}}(v) + \int_{\{\ell \leqq |v| < \ell+k\}} \xi A(x,t,\nabla u).\nabla v \leqq \int_{\{\ell \leqq |v| < \ell+k\}} \xi \nabla v.g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \tag{4.20}$$

Next, we take $\xi \equiv 1$. We verify that there exists $c = c(p)$ such that

$$A(x,t,\nabla u).\nabla v - \nabla v.g \geqq \frac{c_1}{4}(|\nabla u|^p + |\nabla v|^p) - c(|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

where c_1 is the constant in (1.2). Hence (4.16) follows. Thus, from (4.20) and the Hölder inequality, we get, with another constant c, for any $\xi \in C^1([0,T])$ with values in $[0,1]$,

$$-\int_Q \xi_t \overline{T_{k,\ell}}(v) \leqq ckM$$

Thus $\int_\Omega \overline{T_{k,\ell}}(v)(t) \leqq ckM$, for *a.e.* $t \in (0,T)$. We deduce (4.17) by taking $k = 1, \ell = 0$, since $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geqq |r| - 1$, for any $r \in \mathbb{R}$.

Next, from the Gagliardo-Nirenberg embedding Theorem, we have

$$\int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leqq C_1 \|v\|^{\frac{p}{N}}_{L^\infty((0,T);L^1(\Omega))} \int_Q |\nabla T_k(v)|^p,$$

where $C_1 = C_1(N,p)$. Then, from (4.16) and (4.17), we get, for any $k \geqq 1$,

$$\text{meas}\{|v| > k\} \leqq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(v)|^{\frac{p(N+1)}{N}} \leqq C \|v\|^{\frac{p}{N}}_{L^\infty((0,T);L^1(\Omega))} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(v)|^p \leqq C_2 M_1 k^{-p_c},$$

with $C_2 = C_2(N,p,c_1,c_2)$. We obtain

$$\begin{aligned}
\text{meas}\{|\nabla v| > k\} &\leqq \frac{1}{k^p}\int_0^{k^p} \text{meas}(\{|\nabla v|^p > s\})\,ds \\
&\leqq \text{meas}\left\{|v| > k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p}\int_0^{k^p} \text{meas}\left(\left\{|\nabla v|^p > s, |v| \leqq k^{\frac{N}{N+1}}\right\}\right) ds \\
&\leqq C_2 M_1 k^{-m_c} + \frac{1}{k^p} \int\limits_{|v| \leqq k^{\frac{N}{N+1}}} |\nabla v|^p \leqq C_2 M_2 k^{-m_c},
\end{aligned}$$

with $C_3 = C_3(N,p,c_1,c_2)$. Furthermore, for any $k \geqq 1$,

$$\text{meas}\{|h| > k\} + \text{meas}\{|\nabla h| > k\} \leqq C_4 k^{-p} \|h\|^p_X\,,$$

where $C_4 = C_4(N,p,c_1,c_2)$. Therefore, we easily get (4.19). ■

Remark 4.9 *If $\mu \in L^1(Q)$ and $a \equiv 0$ in (1.2), then (4.16) holds for all $k > 0$ and the term $|\Omega|$ in inequality (4.17) can be removed where $M = ||u_0||_{1,\Omega} + |\mu|(Q)$. Furthermore, (4.19) is stated as follows:*

$$\text{meas}\{|u| > k\} \leqq C_3 M^{\frac{p+N}{N}} k^{-p_c}, \qquad \text{meas}\{|\nabla u| > k\} \leqq C_4 M^{\frac{N+2}{N+1}} k^{-m_c}, \forall k > 0. \tag{4.21}$$

To see last inequality, we do in the following way:

$$\begin{aligned}\operatorname{meas}\{|\nabla v|>k\} &\leqq \frac{1}{k^p}\int_0^{k^p}\operatorname{meas}(\{|\nabla v|^p>s\})\,ds\\ &\leqq \operatorname{meas}\left\{|v|>M^{\frac{1}{N+1}}k^{\frac{N}{N+1}}\right\}+\frac{1}{k^p}\int_0^{k^p}\operatorname{meas}\left\{|\nabla v|^p>s,|v|\leqq M^{\frac{1}{N+1}}k^{\frac{N}{N+1}}\right\}ds\\ &\leqq C_4M^{\frac{N+2}{N+1}}k^{-m_c}.\end{aligned}$$

Proposition 4.10 *Let* $\{\mu_n\}\subset\mathcal{M}_b(Q)$, *and* $\{u_{0,n}\}\subset L^1(\Omega)$, *with*

$$\sup_n|\mu_n|(Q)<\infty,\ \text{and}\ \sup_n||u_{0,n}||_{1,\Omega}<\infty.$$

Let u_n *be a R-solution of (1.1) with data* $\mu_n=\mu_{n,0}+\mu_{n,s}$ *and* $u_{0,n}$, *relative to a decomposition* (f_n,g_n,h_n) *of* $\mu_{n,0}$, *and* $v_n=u_n-h_n$. *Assume that* $\{f_n\}$ *is bounded in* $L^1(Q)$, $\{g_n\}$ *bounded in* $(L^{p'}(Q))^N$ *and* $\{h_n\}$ *bounded in* X.

Then, up to a subsequence, $\{v_n\}$ *converges a.e. to a function* v, *such that* $T_k(v)\in X$ *and* $v\in L^\sigma((0,T);W_0^{1,\sigma}(\Omega))\cap L^\infty((0,T);L^1(\Omega))$ *for any* $\sigma\in[1,m_c)$. *And*

- *(i)* $\{v_n\}$ *converges to* v *strongly in* $L^\sigma(Q)$ *for any* $\sigma\in[1,m_c)$, *and* $\sup\|v_n\|_{L^\infty((0,T);L^1(\Omega))}<\infty$,
- *(ii)* $\sup_{k>0}\sup_n\frac{1}{k+1}\int_Q|\nabla T_k(v_n)|^p<\infty$,
- *(iii)* $\{T_k(v_n)\}$ *converges to* $T_k(v)$ *wealkly in* X, *for any* $k>0$,
- *(iv)* $\{A(x,t,\nabla(T_k(v_n)+h_n))\}$ *converges to some* F_k *weakly in* $(L^{p'}(Q))^N$.

Proof. Take $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0)=0$. We combine (4.6) with (4.16), and deduce that $\{S(v_n)_t\}$ is bounded in $X'+L^1(Q)$ and $\{S(v_n)\}$ bounded in X. Hence, $\{S(v_n)\}$ is relatively compact in $L^1(Q)$. On the other hand, we choose $S=S_k$ such that $S_k(z)=z$, if $|z|<k$ and $S(z)=2k\operatorname{sign}z$, if $|z|>2k$. Thanks to (4.17), we obtain

$$\begin{aligned}\operatorname{meas}\{|v_n-v_m|>\sigma\} &\leqq \operatorname{meas}\{|v_n|>k\}+\operatorname{meas}\{|v_m|>k\}+\operatorname{meas}\{|S_k(v_n)-S_k(v_m)|>\sigma\}\\ &\leqq \frac{1}{k}(\|v_n\|_{1,Q}+\|v_m\|_{1,Q})+\operatorname{meas}\{|S_k(v_n)-S_k(v_m)|>\sigma\}\\ &\leqq \frac{C}{k}+\operatorname{meas}\{|S_k(v_n)-S_k(v_m)|>\sigma\}. \end{aligned}\tag{4.22}$$

Thus, up to a subsequence $\{u_n\}$ is a Cauchy sequence in measure, and converges *a.e.* in Q to a function u. Thus, $\{T_k(v_n)\}$ converges to $T_k(v)$ weakly in X, since $\sup_n\|T_k(v_n)\|_X<\infty$ for any $k>0$. And $\left\{|\nabla(T_k(v_n)+h_n)|^{p-2}\nabla(T_k(v_n)+h_n)\right\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$. Furthermore, from (4.18), $\{v_n\}$ converges to v strongly in $L^\sigma(Q)$, for any $\sigma<p_c$. ■

5 The convergence theorem

We first recall some properties of the measures, see [49, Lemma 5], [32].

Proposition 5.1 *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$, where μ_s^+ and μ_s^- are concentrated, respectively, on two disjoint sets E^+ and E^- of zero c_p^Q-capacity. Then, for any $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leqq \delta, \qquad \mu_s^-(E^- \backslash K_\delta^-) \leqq \delta,$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$ with values in $[0,1]$, such that $\psi_\delta^+, \psi_\delta^- = 1$ respectively on K_δ^+, K_δ^-, and $supp(\psi_\delta^+) \cap supp(\psi_\delta^-) = \emptyset$, and

$$||\psi_\delta^+||_X + ||(\psi_\delta^+)_t||_{X'+L^1(Q)} \leqq \delta, \qquad ||\psi_\delta^-||_X + ||(\psi_\delta^-)_t||_{X'+L^1(Q)} \leqq \delta.$$

There exist decompositions $(\psi_\delta^+)_t = \left(\psi_\delta^+\right)_t^1 + \left(\psi_\delta^+\right)_t^2$ and $(\psi_\delta^-)_t = \left(\psi_\delta^-\right)_t^1 + \left(\psi_\delta^-\right)_t^2$ in $X' + L^1(Q)$, such that

$$\left\|\left(\psi_\delta^+\right)_t^1\right\|_{X'} \leqq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^+\right)_t^2\right\|_{1,Q} \leqq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^1\right\|_{X'} \leqq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^2\right\|_{1,Q} \leqq \frac{\delta}{3}. \tag{5.1}$$

Both $\left\{\psi_\delta^+\right\}$ and $\left\{\psi_\delta^-\right\}$ converge to 0, *-weakly in $L^\infty(Q)$, and strongly in $L^1(Q)$ and up to subsequences, a.e. in Q, as δ tends to 0.

Moreover if ρ_n and η_n are as in Theorem 2.1, we have, for any $\delta, \delta_1, \delta_2 > 0$,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n,\delta), \qquad \int_Q \psi_\delta^- d\mu_s^+ \leqq \delta, \qquad \int_Q \psi_\delta^+ d\mu_s^- \leqq \delta, \tag{5.2}$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n,\delta_1,\delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leqq \delta_1 + \delta_2, \tag{5.3}$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n,\delta_1,\delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leqq \delta_1 + \delta_2. \tag{5.4}$$

Hereafter, if $n, \varepsilon, ..., \nu$ are real numbers, and a function ϕ depends on $n, \varepsilon, ..., \nu$ and eventual other parameters $\alpha, \beta, .., \gamma$, and $n \to n_0, \varepsilon \to \varepsilon_0, .., \nu \to \nu_0$, we write $\phi = \omega(n, \varepsilon, .., \nu)$, then this means $\overline{\lim}_{\nu\to\nu_0} .. \overline{\lim}_{\varepsilon\to\varepsilon_0} \overline{\lim}_{n\to n_0} |\phi| = 0$, when the parameters $\alpha, \beta, .., \gamma$ are fixed**.** In the same way, $\phi \leqq \omega(n, \varepsilon, \delta, ..., \nu)$ means $\overline{\lim}_{\nu\to\nu_0} .. \overline{\lim}_{\varepsilon\to\varepsilon_0} \overline{\lim}_{n\to n_0} \phi \leqq 0$, and $\phi \geqq \omega(n, \varepsilon, .., \nu)$ means $-\phi \leqq \omega(n, \varepsilon, .., \nu)$.

Remark 5.2 *In the sequel we use a convergence property, consequence of the Dunford-Pettis theorem, still used in [32]: If $\{a_n\}$ is a sequence in $L^1(Q)$ converging to a weakly in $L^1(Q)$ and $\{b_n\}$ a bounded sequence in $L^\infty(Q)$ converging to b, a.e. in Q, then $\lim_{n\to\infty} \int_Q a_n b_n = \int_Q ab$.*

Next we prove Thorem 2.1.

Scheme of the proof. Let $\{\mu_n\}, \{u_{0,n}\}$ and $\{u_n\}$ satisfying the assumptions of Theorem 2.1. Then we can apply Proposition 4.10. Setting $v_n = u_n - h_n$, up to subsequences, $\{u_n\}$ converges *a.e.* in Q to some function u, and $\{v_n\}$ converges *a.e.* to $v = u - h$, such that $T_k(v) \in X$ and $v \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0,T); L^1(\Omega))$ for every $\sigma \in [1, m_c)$. And $\{v_n\}$ satisfies the conclusions (i) to (iv) of Proposition 4.10. We have

$$\begin{aligned}\mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-,\end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \tag{5.5}$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathcal{M}_b^+(Q) \cap \mathcal{M}_0(Q), \quad \text{and} \quad \rho_n \geqq \rho_{n,0}, \quad \eta_n \geqq \eta_{n,0}. \tag{5.6}$$

Let E^+, E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. For any $\delta_1, \delta_2 > 0$, let $\psi_{\delta_1}^+, \psi_{\delta_2}^+$ and $\psi_{\delta_1}^-, \psi_{\delta_2}^-$ as in Proposition 5.1 and set

$$\Phi_{\delta_1,\delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near E

$$I_1 := \int_{\{|v_n| \leqq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \left(v_n - \langle T_k(v) \rangle_\nu\right) \leqq \omega(n,\nu,\delta_1,\delta_2), \tag{5.7}$$

and far from E,

$$I_2 := \int_{\{|v_n| \leqq k\}} (1 - \Phi_{\delta_1,\delta_2}) A(x,t,\nabla u_n).\nabla (v_n - \langle T_k(v) \rangle_\nu) \leqq \omega(n,\nu,\delta_1,\delta_2). \tag{5.8}$$

Then it follows that

$$\overline{\lim}_{n,\nu} \int_{\{|v_n| \leqq k\}} A(x,t,\nabla u_n).\nabla \left(v_n - \langle T_k(v) \rangle_\nu\right) \leqq 0, \tag{5.9}$$

which implies

$$\overline{\lim}_{n\to\infty} \int_{\{|v_n| \leqq k\}} A(x,t,\nabla u_n).\nabla \left(v_n - T_k(v)\right) \leqq 0, \tag{5.10}$$

since $\{\langle T_k(v) \rangle_\nu\}$ converges to $T_k(v)$ in X. On the other hand, from the weak convergence of $\{T_k(v_n)\}$ to $T_k(v)$ in X, we verify that

$$\int_{\{|v_n| \leqq k\}} A(x,t,\nabla (T_k(v) + h_n)).\nabla \left(T_k(v_n) - T_k(v)\right) = \omega(n).$$

Thus we get

$$\int\limits_{\{|v_n|\leqq k\}} \left(A(x,t,\nabla u_n) - A(x,t,\nabla(T_k(v)+h_n))\right).\nabla\left(u_n - (T_k(v)+h_n)\right) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \qquad a.e. \text{ in } Q. \tag{5.11}$$

Therefore, $\{A(x,t,\nabla u_n)\}$ converges to $A(x,t,\nabla u)$ weakly in $(L^{p'}(Q))^N$; and from (5.10) we find

$$\overline{\lim}_{n\to\infty} \int_Q A(x,t,\nabla u_n).\nabla T_k(v_n) \leqq \int_Q A(x,t,\nabla u)\nabla T_k(v).$$

Otherwise, $\{A(x,t,\nabla(T_k(v_n)+h_n))\}$ converges weakly in $(L^{p'}(Q))^N$to some F_k, from Proposition 4.10, and we obtain that $F_k = A(x,t,\nabla(T_k(v)+h))$. Hence

$$\begin{aligned}\overline{\lim}_{n\to\infty} \int_Q A(x,t,\nabla(T_k(v_n)+h_n)).\nabla(T_k(v_n)+h_n) &\leqq \overline{\lim}_{n\to\infty} \int_Q A(x,t,\nabla u_n).\nabla T_k(v_n) \\ &+ \overline{\lim}_{n\to\infty} \int_Q A(x,t,\nabla(T_k(v_n)+h_n)).\nabla h_n \\ &\leqq \int_Q A(x,t,\nabla(T_k(v)+h)).\nabla(T_k(v)+h).\end{aligned}$$

As a consequence

$$\{T_k(v_n)\} \text{ converges to } T_k(v), \text{ strongly in } X, \qquad \forall k > 0. \tag{5.12}$$

Then *to finish the proof we have to check that u is a solution of* (1.1). ■

In order to prove (5.7) we need a first Lemma, inspired of [32, Lemma 6.1], extending [49, Lemma 6 and Lemma 7]:

Lemma 5.3 *Let $\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)$ be uniformly bounded in $W^{1,\infty}(Q)$ with values in $[0,1]$, such that$\int_Q \psi_{1,\delta} d\mu_s^- \leqq \delta$ and $\int_Q \psi_{2,\delta} d\mu_s^+ \leqq \delta$. Then, under the assumptions of Theorem 2.1,*

$$\frac{1}{m} \int\limits_{\{m\leqq v_n<2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n,m,\delta), \qquad \frac{1}{m} \int\limits_{\{m\leqq v_n<2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n,m,\delta), \tag{5.13}$$

$$\frac{1}{m} \int\limits_{-2m<v_n\leqq -m} |\nabla u_n|^p \psi_{1,\delta} = \omega(n,m,\delta), \qquad \frac{1}{m} \int\limits_{-2m<v_n\leqq -m} |\nabla v_n|^p \psi_{1,\delta} = \omega(n,m,\delta), \tag{5.14}$$

and for any $k > 0$,

$$\int\limits_{\{m\leqq v_n<m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n,m,\delta), \qquad \int\limits_{\{m\leqq v_n<m+k\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n,m,\delta), \tag{5.15}$$

$$\int_{\{-m-k<v_n\leqq -m\}} |\nabla u_n|^p\psi_{1,\delta} = \omega(n,m,\delta), \qquad \int_{\{-m-k<v_n\leqq -m\}} |\nabla v_n|^p\psi_{1,\delta} = \omega(n,m,\delta). \tag{5.16}$$

Proof. (i) Proof of (5.13), (5.14). Set for any $r\in\mathbb{R}$ and any $m,\ell\geqq 1$

$$S_{m,\ell}(r) = \int_0^r \left(\frac{-m+\tau}{m}\chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m+2h-\tau}{2m+\ell}\chi_{(2m+\ell,4m+2h]}(\tau)\right)d\tau,$$

$$S_m(r) = \int_0^r \left(\frac{-m+\tau}{m}\chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau)\right)d\tau.$$

Note that $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi,J,S) = (\psi_{2,\delta},T_1,S_{m,\ell})$ as test functions in (4.15) for u_n, and observe that, from (5.5),

$$\widehat{\mu_{n,0}} = \mu_{n,0} - (h_n)_t = \widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \tag{5.17}$$

Thus we can write $\sum_{i=1}^{6} A_i \leqq \sum_{i=7}^{12} A_i$, where

$$A_1 = -\int_\Omega \psi_{2,\delta}(0)T_1(S_{m,\ell}(u_{0,n}))S_{m,\ell}(u_{0,n}), \quad A_2 = -\int_Q (\psi_{2,\delta})_t\overline{T_1}(S_{m,\ell}(v_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))A(x,t,\nabla u_n)\nabla\psi_{2,\delta},$$

$$A_4 = \int_Q S'_{m,\ell}(v_n)^2\psi_{2,\delta}T'_1(S_{m,\ell}(v_n))A(x,t,\nabla u_n)\nabla v_n,$$

$$A_5 = \frac{1}{m}\int_{\{m\leqq v_n\leqq 2m\}} \psi_{2,\delta}T_1(S_{m,\ell}(v_n))A(x,t,\nabla u_n)\nabla v_n,$$

$$A_6 = -\frac{1}{2m+\ell}\int_{\{2m+\ell\leqq v_n<2(2m+\ell)\}} \psi_{2,\delta}A(x,t,\nabla u_n)\nabla v_n,$$

$$A_7 = \int_Q S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))\psi_{2,\delta}f_n, \qquad A_8 = \int_Q S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))g_n.\nabla\psi_{2,\delta},$$

$$A_9 = \int_Q \left(S'_{m,\ell}(v_n)\right)^2 T'_1(S_{m,\ell}(v_n))\psi_{2,\delta}g_n.\nabla v_n, \qquad A_{10} = \frac{1}{m}\int_{m\leqq v_n\leqq 2m} T_1(S_{m,\ell}(v_n))\psi_{2,\delta}g_n.\nabla v_n,$$

$$A_{11} = -\frac{1}{2m+\ell}\int_{\{2m+\ell\leqq v_n<2(2m+\ell)\}} \psi_{2,\delta}g_n.\nabla v_n, \quad A_{12} = \int_Q S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))\psi_{2,\delta}d\left(\rho_{n,0}-\eta_{n,0}\right).$$

Since $||S_{m,\ell}(u_{0,n})||_{1,\Omega} \leqq \int_{\{m\leqq u_{0,n}\}} u_{0,n}dx$, we find $A_1 = \omega(\ell,n,m)$. Otherwise

$$|A_2| \leqq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)}\int_{\{m\leqq v_n\}} v_n, \qquad |A_3| \leqq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)}\int_{\{m\leqq v_n\}} \left(|a| + c_2|\nabla u_n|^{p-1}\right),$$

which implies $A_2 = \omega(\ell, n, m)$ and $A_3 = \omega(\ell, n, m)$. Using (4.3) for u_n, we have

$$A_6 = -\int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence $A_6 = \omega(\ell, n, m, \delta)$, since $(\rho_{n,s} - \eta_{n,s})^+$ converges to μ_s^+ as $n \to \infty$ in the narrow topology, and $\int_Q \psi_{2,\delta} d\mu_s^+ \leqq \delta$. We also obtain $A_{11} = \omega(\ell)$ from (4.10).

Now $\left\{S'_{m,\ell}(v_n)T_1(S_{m,\ell}(v_n))\right\}_\ell$ converges to $S'_m(v_n)T_1(S_m(v_n))$, $\left\{S'_m(v_n)T_1(S_m(v_n))\right\}_n$ converges to $S'_m(v)\ T_1(S_m(v))$, $\left\{S'_m(v)T_1(S_m(v))\right\}_m$ converges to 0, $*$-weakly in $L^\infty(Q)$, and $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 5.2, we obtain

$$A_7 = \int_Q S'{}_m(v_n)T_1(S_m(v_n))\psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'{}_m(v)T_1(S_m(v))\psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m),$$

$$A_8 = \int_Q S'{}_m(v_n)T_1(S_m(v_n))g_n.\nabla\psi_{2,\delta} + \omega(\ell) = \int_Q S'{}_m(v)T_1(S_m(v))g\nabla\psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m).$$

Otherwise, $A_{12} \leqq \int_Q \psi_{2,\delta} d\rho_n$, and $\left\{\int_Q \psi_{2,\delta} d\rho_n\right\}$ converges to $\int_Q \psi_{2,\delta} d\mu_s^+$, thus $A_{12} \leqq \omega(\ell, n, m, \delta)$. Using Holder inequality and the condition (1.2) we have

$$g_n.\nabla v_n - A(x, t, \nabla u_n)\nabla v_n \le C_1 \left(|g_n|^{p'} + |\nabla h_n|^p + |a|^{p'}\right)$$

with $C_1 = C_1(p, c_2)$, which implies

$$A_9 - A_4 \leqq C_1 \int_Q \left(S'{}_{m,\ell}(v_n)\right)^2 T'_1(S_{m,\ell}(v_n))\psi_{2,\delta} \left(|g_n|^{p'} + |h_n|^p + |a|^{p'}\right) = \omega(\ell, n, m).$$

Similarly we also show that $A_{10} - A_5/2 \leqq \omega(\ell, n, m)$. Combining the estimates, we get $A_5/2 \leqq \omega(\ell, n, m, \delta)$. Using Holder inequality we have

$$A(x, t, \nabla u_n)\nabla v_n \ge \frac{c_1}{2}|\nabla u_n|^p - C_2(|a|^{p'} + |\nabla h_n|^p).$$

with $C_2 = C_2(p, c_1, c_2)$, which implies

$$\frac{1}{m} \int\limits_{\{m \leqq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that for all $m > 4$, $S_{m,\ell}(r) \geqq 1$ for any $r \in [\frac{3}{2}m, 2m]$; hence $T_1(S_{m,\ell}(r) = 1$. So,

$$\frac{1}{m} \int\limits_{\left\{\frac{3}{2}m \leqq v_n < 2m\right\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since $|\nabla v_n|^p \leqq 2^{p-1}|\nabla u_n|^p + 2^{p-1}|\nabla h_n|^p$, there also holds

$$\frac{1}{m} \int_{\left\{\frac{3}{2}m \leqq v_n < 2m\right\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (5.13) by summing on each set $\left\{(\frac{4}{3})^\nu m \leqq v_n \leqq (\frac{4}{3})^{\nu+1} m\right\}$ for $\nu = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$ as test functions in (4.15) for u_n, where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (5.14).

(ii) Proof of (5.15), (5.16). We set, for any $k, m, \ell \geqq 1$,

$$S_{k,m,\ell}(r) = \int_0^r \left(T_k(\tau - T_m(\tau))\chi_{[m,k+m+\ell]} + k\frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell))]} \right) d\tau$$

$$S_{k,m}(r) = \int_0^s T_k(\tau - T_m(\tau))\chi_{[m,\infty)} d\tau.$$

We choose $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$ as test functions in (4.15) for u_n. In the same way we also obtain

$$\int_{\{m \leqq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(v_n)) = \omega(\ell, n, m, \delta).$$

Note that $T_1(S_{k,m,\ell}(r)) = 1$ for any $r \geqq m+1$, thus $\int_{\{m+1 \leqq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$, which implies (5.15) by changing m into $m-1$. Similarly, we obtain (5.16). ■

Next we look at the behaviour near E.

Lemma 5.4 *Estimate (5.7) holds.*

Proof. There holds

$$I_1 = \int_Q \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla T_k(v_n) - \int_{\{|v_n| \leqq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \langle T_k(v) \rangle_\nu.$$

From Proposition 4.10, (iv), $\{A(x,t,\nabla(T_k(v_n) + h_n)).\nabla \langle T_k(v) \rangle_\nu\}$ converges weakly in $L^1(Q)$ to $F_k \nabla \langle T_k(v) \rangle_\nu$. And $\left\{\chi_{\{|v_n| \leqq k\}}\right\}$ converges to $\chi_{|v| \leqq k}$, *a.e.* in Q , and Φ_{δ_1,δ_2} converges to 0 *a.e.* in Q as $\delta_1 \to 0$, and Φ_{δ_1,δ_2} takes its values in $[0,1]$. Thanks to Remark 5.2, we have

$$\begin{aligned}
&\int_{\{|v_n| \leqq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla \langle T_k(v) \rangle_\nu \\
&= \int_Q \chi_{\{|v_n| \leqq k\}} \Phi_{\delta_1,\delta_2} A(x,t,\nabla(T_k(v_n) + h_n)).\nabla \langle T_k(v) \rangle_\nu \\
&= \int_Q \chi_{|v| \leqq k} \Phi_{\delta_1,\delta_2} F_k.\nabla \langle T_k(v) \rangle_\nu + \omega(n) = \omega(n, \nu, \delta_1).
\end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1,\delta_2} A(x,t,\nabla u_n).\nabla T_k(v_n) \leqq \omega(n,\delta_1,\delta_2), \tag{5.18}$$

then we deduce (5.7). As noticed in [32], [49], it is precisely for this estimate that we need the double cut $\psi_{\delta_1}^+\psi_{\delta_2}^+$. To do this, we set, for any $m>k>0$, and any $r\in\mathbb{R}$,

$$\hat{S}_{k,m}(r)=\int_0^r (k-T_k(\tau))\, H_m(\tau)d\tau,$$

where H_m is defined at (4.14). Hence $\mathrm{supp}\hat{S}_{k,m}\subset[-2m,k]$; and $\hat{S}''{}_{k,m}=-\chi_{[-k,k]}+\frac{2k}{m}\chi_{[-2m,-m]}$. We choose $(\varphi,S)=(\psi_{\delta_1}^+\psi_{\delta_2}^+,\hat{S}_{k,m})$ as test functions in (4.2). From (5.17), we can write

$$A_1+A_2-A_3+A_4+A_5+A_6=0,$$

where

$$A_1=-\int_Q (\psi_{\delta_1}^+\psi_{\delta_2}^+)_t\hat{S}_{k,m}(v_n),\quad A_2=\int_Q (k-T_k(v_n))H_m(v_n)A(x,t,\nabla u_n).\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+),$$

$$A_3=\int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+A(x,t,\nabla u_n).\nabla T_k(v_n),\quad A_4=\frac{2k}{m}\int\limits_{\{-2m<v_n\leqq -m\}}\psi_{\delta_1}^+\psi_{\delta_2}^+A(x,t,\nabla u_n).\nabla v_n,$$

$$A_5=-\int_Q (k-T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+d\widehat{\lambda_{n,0}},\quad A_6=\int_Q (k-T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+d\left(\eta_{n,0}-\rho_{n,0}\right);$$

and we estimate A_3. As in [49, p.585], since $\left\{\hat{S}_{k,m}(v_n)\right\}$ converges to $\hat{S}_{k,m}(v)$ weakly in X, and $\hat{S}_{k,m}(v)\in L^\infty(Q)$, and from (5.1), there holds

$$A_1=-\int_Q (\psi_{\delta_1}^+)_t\psi_{\delta_2}^+\hat{S}_{k,m}(v)-\int_Q \psi_{\delta_1}^+(\psi_{\delta_2}^+)_t\hat{S}_{k,m}(v)+\omega(n)=\omega(n,\delta_1).$$

Next consider A_2. Notice that $v_n=T_{2m}(v_n)$ on $\mathrm{supp}(H_m(v_n))$. From Proposition 4.10, (iv), the sequence $\left\{A(x,t,\nabla\left(T_{2m}(v_n)+h_n\right)).\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)\right\}$ converges to $F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)$ weakly in $L^1(Q)$. Thanks to Remark 5.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ in X to 0 as δ_1 tends to 0, we find

$$A_2=\int_Q (k-T_k(v))H_m(v)F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)+\omega(n)=\omega(n,\delta_1).$$

Then consider A_4. Then for some $C=C(p,c_2)$,

$$|A_4|\leqq C\frac{2k}{m}\int\limits_{\{-2m<v_n\leqq -m\}}\left(|\nabla u_n|^p+|\nabla v_n|^p+|a|^{p'}\right)\psi_{\delta_1}^+\psi_{\delta_2}^+.$$

Since $\psi_{\delta_1}^+$ takes its values in $[0,1]$, from Lemma 5.3, we get in particular $A_4=\omega(n,\delta_1,m,\delta_2)$.

Now estimate A_5. The sequence $\left\{(k-T_k(v_n))H_m(v_n)\psi^+_{\delta_1}\psi^+_{\delta_2}\right\}$ converges weakly in X to $(k-T_k(v))H_m(v)\psi^+_{\delta_1}\psi^+_{\delta_2}$, and $\{(k-T_k(v_n))H_m(v_n)\}$ converges *-weakly in $L^\infty(Q)$ and *a.e.* in Q to $(k-T_k(v))H_m(v)$. Otherwise $\{f_n\}$ converges to f weakly in $L^1(Q)$ and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. Thanks to Remark 5.2 and the convergence of $\psi^+_{\delta_1}\psi^+_{\delta_2}$ to 0 in X and *a.e.* in Q as $\delta_1 \to 0$, we deduce that

$$A_5 = -\int_Q (k-T_k(v_n))H_m(v)\psi^+_{\delta_1}\psi^+_{\delta_2}d\widehat{\nu_0} + \omega(n) = \omega(n,\delta_1),$$

where $\widehat{\nu_0} = f - \operatorname{div} g$.

Finally $A_6 \leqq 2k\int_Q \psi^+_{\delta_1}\psi^+_{\delta_2}d\eta_n$; using (5.2) we also find $A_6 \leqq \omega(n,\delta_1,m,\delta_2)$. By addition, since A_3 does not depend on m, we obtain

$$A_3 = \int_Q \psi^+_{\delta_1}\psi^+_{\delta_2}A(x,t,\nabla u_n)\nabla T_k(v_n) \leqq \omega(n,\delta_1,\delta_2).$$

Reasoning as before with $(\psi^-_{\delta_1}\psi^-_{\delta_2}, \check{S}_{k,m})$ as test function in (4.2), where $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q \psi^-_{\delta_1}\psi^-_{\delta_2}A(x,t,\nabla u_n)\nabla T_k(v_n) \leqq \omega(n,\delta_1,\delta_2).$$

Then, (5.18) holds. ■

Next we look at the behaviour far from E.

Lemma 5.5 . *Estimate (5.8) holds.*

Proof. Here we estimate I_2; we can write

$$I_2 = \int_{\{|v_n|\leqq k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla\left(T_k(v_n) - \langle T_k(v)\rangle_\nu\right).$$

Following the ideas of [51], used also in [49], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v_n - T_k\left(v_n\right)\right).$$

Recall that $\|\langle T_k(v)\rangle_\nu\|_{\infty,Q} \leqq k$, and observe that

$$R_{n,\nu,\ell} = 2k\,\mathrm{sign}(v_n) \quad \text{in } \{|v_n| \geqq \ell+2k\}, \qquad |R_{n,\nu,\ell}| \leqq 4k, \qquad R_{n,\nu,\ell} = \omega(n,\nu,\ell) \text{ a.e. in } Q, \tag{5.19}$$

$$\lim_{n\to\infty} R_{n,\nu,\ell} = T_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v - T_k\left(v\right)\right), \qquad \text{a.e. in } Q, \text{ and weakly in } X. \tag{5.20}$$

Next consider $\xi_{1,n_1} \in C_c^\infty([0,T)), \xi_{2,n_2} \in C_c^\infty((0,T])$ with values in $[0,1]$, such that $(\xi_{1,n_1})_t \leqq 0$ and $(\xi_{2,n_2})_t \geqq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{1,n_2}(t)\}$) converges to $1,$ for any $t \in [0,T)$ (resp. $t \in (0,T]$);

and moreover, for any $a \in C([0,T];L^1(\Omega))$, $\left\{\int_Q a(\xi_{1,n_1})_t\right\}$ and $\int_Q a(\xi_{2,n_2})_t$ converge respectively to $-\int_\Omega a(T,.)$ and $\int_\Omega a(0,.)$. We set

$$\varphi = \varphi_{n,n_1,n_2,l_1,l_2,\ell} = \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})[T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1} - \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})[T_{\ell-k}\left(v_n - T_k(v_n)\right)]_{-l_2}.$$

We can see that

$$\varphi - (1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} = \omega(l_1,l_2,n_1,n_2) \qquad \text{in norm in } X \text{ and } a.e. \text{ in } Q. \tag{5.21}$$

We can choose $(\varphi, S) = (\varphi_{n,n_1,n_2,l_1,l_2,\ell}, \overline{H_m})$ as test functions in (4.7) for u_n, where $\overline{H_m}$ is defined at (4.14), with $m > \ell + 2k$. We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$\begin{aligned}
&A_1 = \int_\Omega \varphi(T)\overline{H_m}(v_n(T))dx, \qquad A_2 = -\int_\Omega \varphi(0)\overline{H_m}(u_{0,n})dx,\\
&A_3 = -\int_Q \varphi_t \overline{H_m}(v_n), \qquad A_4 = \int_Q H_m(v_n)A(x,t,\nabla u_n).\nabla\varphi,\\
&A_5 = \int_Q \varphi H'{}_m(v_n)A(x,t,\nabla u_n).\nabla v_n, \qquad A_6 = \int_Q H_m(v_n)\varphi d\widehat{\lambda_{n,0}},\\
&A_7 = \int_Q H_m(v_n)\varphi d\left(\rho_{n,0} - \eta_{n,0}\right).
\end{aligned}$$

Estimate of A_4. This term allows to study I_2. Indeed, $\{H_m(v_n)\}$ converges to 1, $a.e.$ in Q; thanks to (5.21), (5.19) (5.20), we have

$$\begin{aligned}
A_4 &= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} - \int_Q R_{n,\nu,\ell}A(x,t,\nabla u_n).\nabla\Phi_{\delta_1,\delta_2} + \omega(l_1,l_2,n_1,n_2,m)\\
&= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell)\\
&= I_2 + \int_{\{|v_n|>k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell)\\
&= I_2 + B_1 + B_2 + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell),
\end{aligned}$$

where

$$\begin{aligned}
&B_1 = \int_{\{|v_n|>k\}} (1-\Phi_{\delta,\eta})(\chi_{\left|v_n - \langle T_k(v)\rangle_\nu\right| \leqq \ell+k} - \chi_{||v_n|-k| \leqq \ell-k})A(x,t,\nabla u_n).\nabla v_n,\\
&B_2 = -\int_{\{|v_n|>k\}} (1-\Phi_{\delta_1,\delta_2})\chi_{\left|v_n - \langle T_k(v)\rangle_\nu\right| \leqq \ell+k}A(x,t,\nabla u_n).\nabla\langle T_k(v)\rangle_\nu.
\end{aligned}$$

Now $\{A(x,t,\nabla\left(T_{\ell+2k}(v_n)+h_n\right)).\nabla\langle T_k(v)\rangle_\nu\}$ converges to $F_{\ell+2k}\nabla\langle T_k(v)\rangle_\nu$, weakly in $L^1(Q)$. Otherwise $\left\{\chi_{|v_n|>k}\chi_{\left|v_n-\langle T_k(v)\rangle_\nu\right|\leqq\ell+k}\right\}$ converges to $\chi_{|v|>k}\chi_{\left|v-\langle T_k(v)\rangle_\nu\right|\leqq\ell+k}$, *a.e.* in Q. And $\{\langle T_k(v)\rangle_\nu\}$ converges to $T_k(v)$ strongly in X. Thanks to Remark 5.2 we get

$$\begin{aligned}B_2&=-\int_Q\left(1-\Phi_{\delta_1,\delta_2}\right)\chi_{|v|>k}\,\chi_{\left|v-\langle T_k(v)\rangle_\nu\right|\leqq\ell+k}F_{\ell+2k}.\nabla\langle T_k(v)\rangle_\nu+\omega(n)\\&=-\int_Q\left(1-\Phi_{\delta_1,\delta_2}\right)\chi_{|v|>k}\,\chi_{|v-T_k(v)|\leqq\ell+k}F_{\ell+2k}.\nabla T_k(v)+\omega(n,\nu)=\omega(n,\nu),\end{aligned}$$

since $\nabla T_k(v)\ \chi_{|v|>k}=0$. Besides, we see that, for some $C=C(p,c_2)$,

$$|B_1|\leqq C\int_{\{\ell-2k\leqq|v_n|<\ell+2k\}}\left(1-\Phi_{\delta_1,\delta_2}\right)\left(|\nabla u_n|^p+|\nabla v_n|^p+|a|^{p'}\right).$$

Using (5.3) and (5.4) and applying (5.15) and (5.16) to $1-\Phi_{\delta_1,\delta_2}$, we obtain, for $k>0$

$$\int_{\{m\leqq|v_n|<m+4k\}}(|\nabla u_n|^p+|\nabla v_n|^p)(1-\Phi_{\delta_1,\delta_2})=\omega(n,m,\delta_1,\delta_2). \tag{5.22}$$

Thus, $B_1=\omega(n,\nu,\ell,\delta_1,\delta_2)$, hence $B_1+B_2=\omega(n,\nu,\ell,\delta_1,\delta_2)$. Then

$$A_4=I_2+\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2). \tag{5.23}$$

Estimate of A_5. For $m>\ell+2k$, since $|\varphi|\leqq 2\ell$, and (5.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned}A_5&=\int_Q(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}H_m^{'}(v_n)A(x,t,\nabla u_n).\nabla v_n+\omega(l_1,l_2,n_1,n_2)\\&=-\frac{2k}{m}\int_{\{m\leqq|v_n|<2m\}}(1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla v_n+\omega(l_1,l_2,n_1,n_2);\end{aligned}$$

here, the final equality followed from the relation, since $m>\ell+2k$,

$$R_{n,\nu,\ell}H_m^{'}(v_n)=-\frac{2k}{m}\chi_{m\leqq|v_n|\leqq 2m},\quad a.e.\text{ in }Q. \tag{5.24}$$

Next we go to the limit in m, by using (4.3), (4.4) for u_n, with $\phi=(1-\Phi_{\delta_1,\delta_2})$. There holds

$$A_5=-2k\int_Q(1-\Phi_{\delta_1,\delta_2})d\left((\rho_{n,s}-\eta_{n,s})^{+}+(\rho_{n,s}-\eta_{n,s})^{-}\right)+\omega(l_1,l_2,n_1,n_2,m).$$

Then, from (5.3) and (5.4), we get $A_5=\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of A_6. Again, from (5.21),

$$A_6 = \int_Q H_m(v_n)\varphi f_n + \int_Q g_n.\nabla(H_m(v_n)\varphi)$$
$$= \int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}f_n + \int_Q g_n.\nabla(H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}) + \omega(l_1,l_2,n_1,n_2).$$

Thus we can write $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1,l_2,n_1,n_2)$, where

$$D_1 = \int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}f_n, \qquad D_2 = \int_Q (1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}H'_m(v_n)g_n.\nabla v_n,$$
$$D_3 = \int_Q H_m(v_n)(1-\Phi_{\delta_1,\delta_2})g_n.\nabla R_{n,\nu,\ell}, \qquad D_4 = -\int_Q H_m(v_n)R_{n,\nu,\ell}g_n.\nabla\Phi_{\delta_1,\delta_2}.$$

Since $\{f_n\}$ converges to f weakly in $L^1(Q)$, and (5.19)-(5.20) hold, we get from Remark 5.2,

$$D_1 = \int_Q (1-\Phi_{\delta_1,\delta_2})\left(T_{\ell+k}\left(v-\langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v-T_k\left(v\right)\right)\right)f + \omega(m,n) = \omega(m,n,\nu,\ell).$$

We deduce from (4.10) that $D_2 = \omega(m)$. Next consider D_3. Note that $H_m(v_n) = 1 + \omega(m)$, and (5.20) holds, and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, and $\langle T_k(v)\rangle_\nu$ converges to $T_k(v)$ strongly in X. Then we obtain successively that

$$D_3 = \int_Q (1-\Phi_{\delta_1,\delta_2})g.\nabla\left(T_{\ell+k}\left(v-\langle T_k(v)\rangle_\nu\right) - T_{\ell-k}\left(v-T_k\left(v\right)\right)\right) + \omega(m,n)$$
$$= \int_Q (1-\Phi_{\delta_1,\delta_2})g.\nabla\left(T_{\ell+k}\left(v-T_k(v)\right) - T_{\ell-k}\left(v-T_k\left(v\right)\right)\right) + \omega(m,n,\nu)$$
$$= \omega(m,n,\nu,\ell).$$

Similarly we also get $D_4 = \omega(m,n,\nu,\ell)$. Thus $A_6 = \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of A_7. We have

$$|A_7| = \left|\int_Q S'_m(v_n)\left(1-\Phi_{\delta_1,\delta_2}\right)R_{n,\nu,\ell}d\left(\rho_{n,0}-\eta_{n,0}\right)\right| + \omega(l_1,l_2,n_1,n_2)$$
$$\leqq 4k\int_Q \left(1-\Phi_{\delta_1,\delta_2}\right)d\left(\rho_n+\eta_n\right) + \omega(l_1,l_2,n_1,n_2).$$

From (5.3) and (5.4) we get $A_7 = \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of $A_1 + A_2 + A_3$. We set

$$J(r) = T_{\ell-k}\left(r-T_k\left(r\right)\right), \qquad \forall r \in \mathbb{R},$$

and use the notations $\overline{J}$ and$\mathcal{J}$ of (4.11). From the definitions of ξ_{1,n_1}, ξ_{1,n_2}, we can see that

$$\begin{aligned} A_1 + A_2 &= -\int_\Omega J(v_n(T))\overline{H_m}(v_n(T)) - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)\overline{H_m}(u_{0,n}) + \omega(l_1, l_2, n_1, n_2) \\ &= -\int_\Omega J(v_n(T))v_n(T) - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)u_{0,n} + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \tag{5.25}$$

where $z_\nu = \langle T_k(v)\rangle_\nu(0)$. We can write $A_3 = F_1 + F_2$, where

$$F_1 = -\int_Q \Big(\xi_{n_1}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell+k}\,(v_n - \langle T_k(v)\rangle_\nu)]_{l_1}\Big)_t \overline{H_m}(v_n),$$

$$F_2 = \int_Q \big(\xi_{n_2}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell-k}\,(v_n - T_k\,(v_n)))]_{-l_2}\big)_t \overline{H_m}(v_n).$$

Estimate of F_2. We write $F_2 = G_1 + G_2 + G_3$, with

$$G_1 = -\int_Q (\Phi_{\delta_1,\delta_2})_t \xi_{n_2}[T_{\ell-k}\,(v_n - T_k\,(v_n))]_{-l_2}\overline{H_m}(v_n),$$

$$G_2 = \int_Q (1 - \Phi_{\delta_1,\delta_2})(\xi_{n_2})_t[T_{\ell-k}\,(v_n - T_k\,(v_n))]_{-l_2}\overline{H_m}(v_n),$$

$$G_3 = \int_Q \xi_{n_2}(1 - \Phi_{\delta_1,\delta_2})\big([T_{\ell-k}\,(v_n - T_k\,(v_n))]_{-l_2}\big)_t \overline{H_m}(v_n).$$

We find easily

$$G_1 = -\int_Q (\Phi_{\delta_1,\delta_2})_t J(v_n)v_n + \omega(l_1, l_2, n_1, n_2, m),$$

$$G_2 = \int_Q (1 - \Phi_{\delta_1,\delta_2})(\xi_{n_2})_t J(v_n)\overline{H_m}(v_n) + \omega(l_1, l_2) = \int\limits_\Omega J(u_{0,n})u_{0,n} + \omega(l_1, l_2, n_1, n_2, m).$$

Next consider G_3. Setting $b = \overline{H_m}(v_n)$, there holds from (4.13) and (4.12),

$$(([J(b)]_{-l_2})_t b)(.,t) = \frac{b(.,t)}{l_2}(J(b)(.,t) - J(b)(.,t - l_2)).$$

Hence

$$\big([T_{\ell-k}\,(v_n - T_k\,(v_n))]_{-l_2}\big)_t \overline{H_m}(v_n) \geqq \Big(\big[\mathcal{J}(\overline{H_m}(v_n))\big]_{-l_2}\Big)_t = \big([\mathcal{J}(v_n)]_{-l_2}\big)_t,$$

since $\mathcal{J}$ is constant in $\{|r| \geqq m+\ell+2k\}$. Integrating by parts in G_3, we find

$$
\begin{aligned}
G_3 &\geqq \int_Q \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})\big([\mathcal{J}(v_n)]_{-l_2}\big)_t \\
&= -\int_Q (\xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2}))_t[\mathcal{J}(v_n)]_{-l_2} + \int_\Omega \xi_{2,n_2}(T)[\mathcal{J}(v_n)]_{-l_2}(T) \\
&= -\int_Q (\xi_{2,n_2})_t(1-\Phi_{\delta_1,\delta_2})\mathcal{J}(v_n) \\
&+ \int_Q \xi_{2,n_2}(\Phi_{\delta_1,\delta_2})_t\mathcal{J}(v_n) + \int_\Omega \xi_{2,n_2}(T)\mathcal{J}(v_n(T)) + \omega(l_1,l_2) \\
&= -\int_\Omega \mathcal{J}(u_{0,n}) + \int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{J}(v_n) + \int_\Omega \mathcal{J}(v_n(T)) + \omega(l_1,l_2,n_1,n_2).
\end{aligned}
$$

Therefore, since $\mathcal{J}(v_n) - J(v_n)v_n = -\overline{J}(v_n)$ and $\overline{J}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$, we obtain

$$
F_2 \geqq \int_\Omega \overline{J}(u_{0,n}) \ - \int_Q (\Phi_{\delta_1,\delta_2})_t\overline{J}(v_n) + \int_\Omega \mathcal{J}(v_n(T)) + \omega(l_1,l_2,n_1,n_2,m). \tag{5.26}
$$

Estimate of F_1. Since $m > \ell+2k$, there holds $T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) = T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)$ on $\operatorname{supp}\overline{H_m}(v_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$
\begin{aligned}
L_1 &= -\int_Q \Big(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\Big)_t \left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)\rangle_\nu\right) \\
L_2 &= -\int_Q \Big(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\Big)_t \langle T_k(\overline{H_m}(v))\rangle_\nu.
\end{aligned}
$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$
\begin{aligned}
L_2 &= \int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\big(\langle T_k(\overline{H_m}(v))\rangle_\nu\big)_t \\
&+ \int_\Omega \xi_{1,n_1}(0)\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}(0)\langle T_k(\overline{H_m}(v))\rangle_\nu(0) \\
&= \nu\int_Q (1-\Phi_{\delta_1,\delta_2})T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) + \int_\Omega T_{\ell+k}\left(u_{0,n} - z_\nu\right) z_\nu + \omega(l_1,l_2,n_1,n_2).
\end{aligned} \tag{5.27}
$$

We decompose L_1 into $L_1 = K_1 + K_2 + K_3$, where

$$
\begin{aligned}
K_1 &= -\int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right) \\
K_2 &= \int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right) \\
K_3 &= -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\Big(\big[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\big]_{l_1}\Big)_t\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)\rangle_\nu\right).
\end{aligned}
$$

Then we check easily that

$$K_1 = \int\limits_\Omega T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)(T)\left(v_n - \langle T_k(v)\rangle_\nu\right)(T)dx + \omega(l_1, l_2, n_1, n_2, m),$$

$$K_2 = \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)\left(v_n - \langle T_k(v)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m).$$

Next consider K_3. Here we use the function $\mathcal{T}_k$ defined at (4.13). We set $b = \overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu$. Hence from (4.12),

$$\begin{aligned}(([T_{\ell+k}(b)]_{l_1})_t b)(.,t) &= \frac{b(.,t)}{l_1}(T_{\ell+k}(b)(.,t+l_1) - T_{\ell+k}(b)(.,t)) \\ &\leqq \frac{1}{l_1}(\mathcal{T}_{\ell+k}(b)((.,t+l_1)) - \mathcal{T}_{\ell+k}(b)(.,t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t.\end{aligned}$$

Thus

$$\begin{aligned}\left(\left[T_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\right]_{l_1}\right)_t\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right) &\leqq \left(\left[\mathcal{T}_{\ell+k}\left(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v))\rangle_\nu\right)\right]_{l_1}\right)_t \\ &= \left([\mathcal{T}_{\ell+k}(v_n - \langle T_k(v)\rangle_\nu]_{l_1}\right)_t.\end{aligned}$$

Then

$$\begin{aligned}K_3 &\geqq -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\left(\left[\mathcal{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)\right]_{l_1}\right)_t \\ &= \int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})[\mathcal{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1} - \int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t[\mathcal{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1} \\ &+ \int\limits_\Omega \xi_{1,n_1}(0)[\mathcal{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)]_{l_1}(0) \\ &= -\int\limits_\Omega \mathcal{T}_{\ell+k}\left(v_n(T) - \langle T_k(v)\rangle_\nu(T)\right) - \int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) \\ &+ \int\limits_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n} - z_\nu\right) + \omega(l_1, l_2, n_1, n_2).\end{aligned}$$

We find by addition, since $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \overline{T}_{\ell+k}(r)$ for any $r \in \mathbb{R}$,

$$\begin{aligned}L_1 &\geqq \int\limits_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n} - z_\nu\right) + \int\limits_\Omega \overline{T}_{\ell+k}\left(v_n(T) - \langle T_k(v)\rangle_\nu(T)\right) \\ &+ \int_Q (\Phi_{\delta_1,\delta_2})_t\overline{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m).\end{aligned} \tag{5.28}$$

We deduce from (5.28), (5.27), (5.26),

$$A_3 \geqq \int_\Omega \overline{J}(u_{0,n}) + \int_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n}-z_\nu\right) + \int_\Omega T_{\ell+k}\left(u_{0,n}-z_\nu\right) z_\nu \tag{5.29}$$

$$+ \int_\Omega \overline{T}_{\ell+k}\left(v_n(T) - \langle T_k(v)\rangle_\nu(T)\right) + \int_\Omega \mathcal{J}(v_n(T)) + \int_Q (\Phi_{\delta_1,\delta_2})_t \left(\overline{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v_n)\right)$$

$$+ \nu \int_Q (1-\Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) + \omega(l_1,l_2,n_1,n_2,m).$$

Next we add (5.25) and (5.29). Note that $\mathcal{J}(v_n(T)) - J(v_n(T))v_n(T) = -\overline{J}(v_n(T))$, and also $\mathcal{T}_{\ell+k}\left(u_{0,n}-z_\nu\right) - T_{\ell+k}\left(u_{0,n}-z_\nu\right)\left(z_\nu - u_{0,n}\right) = -\overline{T}_{\ell+k}\left(u_{0,n}-z_\nu\right)$. Then we find

$$A_1 + A_2 + A_3 \geqq \int_\Omega \left(\overline{J}(u_{0,n}) - \overline{T}_{\ell+k}\left(u_{0,n}-z_\nu\right)\right) + \int_\Omega \left(\overline{T}_{\ell+k}\left(v_n(T) - \langle T_k(v)\rangle_\nu(T)\right) - \overline{J}(v_n(T))\right)$$

$$+ \int_Q (\Phi_{\delta_1,\delta_2})_t \left(\overline{T}_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v_n)\right)$$

$$+ \nu \int_Q (1-\Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(v_n - \langle T_k(v)\rangle_\nu\right)\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) + \omega(l_1,l_2,n_1,n_2,m).$$

Notice that $\overline{T}_{\ell+k}(r-s) - \overline{J}(r) \geqq 0$ for any $r,s \in \mathbb{R}$ such that $|s| \leqq k$; thus

$$\int_\Omega \left(\overline{T}_{\ell+k}\left(v_n(T) - \langle T_k(v)\rangle_\nu(T)\right) - \overline{J}(v_n(T))\right) \geqq 0.$$

And $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$ and $\{v_n\}$ converges to v in $L^1(Q)$ from Proposition 4.10. Thus we obtain

$$A_1 + A_2 + A_3 \geqq \textstyle\int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) + \int_Q (\Phi_{\delta_1,\delta_2})_t \left(\overline{T}_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v)\right)$$

$$+\nu \textstyle\int_Q (1-\Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right)\left(T_k(v) - \langle T_k(v)\rangle_\nu\right) + \omega(l_1,l_2,n_1,n_2,m,n).$$

Moreover $T_{\ell+k}(r-s)\left(T_k(r) - s\right) \geqq 0$ for any $r,s \in \mathbb{R}$ such that $|s| \leqq k$, hence

$$A_1 + A_2 + A_3 \geqq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) + \int_Q (\Phi_{\delta_1,\delta_2})_t \left(\overline{T}_{\ell+k}\left(v - \langle T_k(v)\rangle_\nu\right) - \overline{J}(v)\right)$$

$$+\omega(l_1,l_2,n_1,n_2,m,n).$$

As $\nu \to \infty$, $\{z_\nu\}$ converges to $T_k(u_0)$, *a.e.* in Ω, thus we get

$$A_1 + A_2 + A_3 \geqq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - T_k(u_0)\right)\right) + \int_Q (\Phi_{\delta_1,\delta_2})_t \left(\overline{T}_{\ell+k}\left(v - T_k(v)\right) - \overline{J}(v)\right)$$

$$+ \omega(l_1,l_2,n_1,n_2,m,n,\nu).$$

Finally $\left|\overline{T}_{\ell+k}\left(r-T_k(r)\right)-\overline{J}(r)\right| \leqq 2k|r|\chi_{\{|r|\geqq\ell\}}$ for any $r\in\mathbb{R}$, thus

$$A_1+A_2+A_3 \geqq \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell).$$

Combining all the estimates, we obtain $I_2 \leqq \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$ which implies (5.8), since I_2 does not depend on l_1,l_2,n_1,n_2,m,ℓ. ■

Next we conclude the proof of Theorem 2.1:

Lemma 5.6 *The function u is a R-solution of (1.1).*

Proof. (i) First show that u satisfies (4.2). Here we proceed as in [49]. Let $\varphi\in X\cap L^\infty(Q)$ such $\varphi_t\in X'+L^1(Q)$, $\varphi(.,T)=0$, and $S\in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on $\mathbb{R}$, $S(0)=0$. Let $M>0$ such that supp$S'\subset[-M,M]$. Taking successively (φ,S) and $(\varphi\psi_\delta^\pm,S)$ as test functions in (4.2) applied to u_n, we can write

$$A_1+A_2+A_3+A_4=A_5+A_6+A_7,\qquad A_{2,\delta,\pm}+A_{3,\delta,\pm}+A_{4,\delta,\pm}=A_{5,\delta,\pm}+A_{6,\delta,\pm}+A_{7,\delta,\pm},$$

where

$$A_1=-\int_\Omega\varphi(0)S(u_{0,n}),\quad A_2=-\int_Q\varphi_tS(v_n),\quad A_{2,\delta,\pm}=-\int_Q(\varphi\psi_\delta^\pm)_tS(v_n),$$
$$A_3=\int_Q S'(v_n)A(x,t,\nabla u_n).\nabla\varphi,\quad A_{3,\delta,\pm}=\int_Q S'(v_n)A(x,t,\nabla u_n).\nabla(\varphi\psi_\delta^\pm),$$

$$A_4=\int_Q S''(v_n)\varphi A(x,t,\nabla u_n).\nabla v_n,\quad A_{4,\delta,\pm}=\int_Q S''(v_n)\varphi\psi_\delta^\pm A(x,t,\nabla u_n).\nabla v_n,$$
$$A_5=\int_Q S'(v_n)\varphi d\widehat{\lambda_{n,0}},\quad A_6=\int_Q S'(v_n)\varphi d\rho_{n,0},\quad A_7=-\int_Q S'(v_n)\varphi d\eta_{n,0},$$
$$A_{5,\delta,\pm}=\int_Q S'(v_n)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}},\quad A_{6,\delta,\pm}=\int_Q S'(v_n)\varphi\psi_\delta^\pm d\rho_{n,0},\quad A_{7,\delta,\pm}=-\int_Q S'(v_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$, and $\{S(v_n)\}$ converges to $S(v)$ strongly in X and weak* in $L^\infty(Q)$, there holds, from (5.2),

$$A_1=-\int_\Omega\varphi(0)S(u_0)+\omega(n),\quad A_2=-\int_Q\varphi_tS(v)+\omega(n),\quad A_{2,\delta,\psi_\delta^\pm}=\omega(n,\delta).$$

Moreover $T_M(v_n)$ converges to $T_M(v)$, then $T_M(v_n)+h_n$ converges to $T_k(v)+h$ strongly in X, thus

$$\begin{aligned}A_3&=\int_Q S'(v_n)A(x,t,\nabla\left(T_M\left(v_n\right)+h_n\right)).\nabla\varphi\\&=\int_Q S'(v)A(x,t,\nabla\left(T_M\left(v\right)+h\right)).\nabla\varphi+\omega(n)\\&=\int_Q S'(v)A(x,t,\nabla u).\nabla\varphi+\omega(n);\end{aligned}$$

and

$$\begin{aligned} A_4 &= \int_Q S''(v_n)\varphi A(x,t,\nabla\left(T_M\left(v_n\right)+h_n\right)).\nabla T_M\left(v_n\right) \\ &= \int_Q S''(v)\varphi A(x,t,\nabla\left(T_M\left(v\right)+h\right)).\nabla T_M\left(v\right)+\omega(n) \\ &= \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v+\omega(n). \end{aligned}$$

In the same way, since $\psi_\delta^\pm$ converges to 0 in X,

$$\begin{aligned} A_{3,\delta,\pm} &= \int_Q S'(v)A(x,t,\nabla u).\nabla(\varphi\psi_\delta^\pm)+\omega(n)=\omega(n,\delta), \\ A_{4,\delta,\pm} &= \int_Q S''(v)\varphi\psi_\delta^\pm A(x,t,\nabla u).\nabla v+\omega(n)=\omega(n,\delta). \end{aligned}$$

And $\{g_n\}$ converges strongly in $(L^{p'}(\Omega))^N$, thus

$$\begin{aligned} A_5 &= \int_Q S'(v_n)\varphi f_n+\int_Q S'(v_n)g_n.\nabla\varphi+\int_Q S''(v_n)\varphi g_n.\nabla T_M(v_n) \\ &= \int_Q S'(v)\varphi f+\int_Q S'(v)g.\nabla\varphi+\int_Q S''(v)\varphi g.\nabla T_M(v)+\omega(n) \\ &= \int_Q S'(v)\varphi d\widehat{\mu_0}+\omega(n). \end{aligned}$$

and $A_{5,\delta,\pm}=\int_Q S'(v)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}}+\omega(n)=\omega(n,\delta)$. Then $A_{6,\delta,\pm}+A_{7,\delta,\pm}=\omega(n,\delta)$. From (5.2) we verify that $A_{7,\delta,+}=\omega(n,\delta)$ and $A_{6,\delta,-}=\omega(n,\delta)$. Moreover, from (5.6) and (5.2), we find

$$|A_6-A_{6,\delta,+}|\leqq\int_Q\left|S'(v_n)\varphi\right|(1-\psi_\delta^+)d\rho_{n,0}\leqq\|S\|_{W^{2,\infty}(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\int_Q(1-\psi_\delta^+)d\rho_n=\omega(n,\delta).$$

Similarly we also have $|A_7-A_{7,\delta,-}|\leqq\omega(n,\delta)$. Hence $A_6=\omega(n)$ and $A_7=\omega(n)$. Therefore, we finally obtain (4.2):

$$-\int_\Omega\varphi(0)S(u_0)-\int_Q\varphi_t S(v)+\int_Q S'(v)A(x,t,\nabla u).\nabla\varphi+\int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v=\int_Q S'(v)\varphi d\widehat{\mu_0}. \tag{5.30}$$

(ii) Next, we prove (4.3) and (4.4). We take $\varphi \in C_c^\infty(Q)$ and take $((1-\psi_\delta^-)\varphi, \overline{H_m})$ as test functions in (5.30), with $\overline{H_m}$ as in (4.14). We can write $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$, where

$$\begin{aligned} &D_{1,m} = -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t \overline{H_m}(v), \qquad D_{2,m} = \int_Q H_m(v) A(x,t,\nabla u).\nabla\left((1-\psi_\delta^-)\varphi\right), \\ &D_{3,m} = \int_Q H_m(v)(1-\psi_\delta^-)\varphi d\widehat{\mu_0}, \qquad D_{4,m} = \frac{1}{m}\int_{m\leqq v\leqq 2m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u).\nabla v, \\ &D_{5,m} = -\frac{1}{m}\int_{-2m\leqq v\leqq -m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u)\nabla v. \end{aligned} \tag{5.31}$$

Taking the same test functions in (4.2) applied to u_n, there holds $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$, where

$$\begin{aligned} &D_{1,m}^n = -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t \overline{H_m}(v_n), \qquad D_{2,m}^n = \int_Q H_m(v_n) A(x,t,\nabla u_n).\nabla\left((1-\psi_\delta^-)\varphi\right), \\ &D_{3,m}^n = \int_Q H_m(v_n)(1-\psi_\delta^-)\varphi d(\widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0}), \quad D_{4,m}^n = \frac{1}{m}\int_{m\leqq v\leqq 2m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u_n).\nabla v_n, \\ &D_{5,m}^n = -\frac{1}{m}\int_{-2m\leqq v_n\leqq -m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u_n).\nabla v_n \end{aligned} \tag{5.32}$$

In (5.32), we go to the limit as $m \to \infty$. Since $\{\overline{H}_m(v_n)\}$ converges to v_n and $\{H_m(v_n)\}$ converges to 1, *a.e.* in Q, and $\{\nabla H_m(v_n)\}$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation $D_1^n + D_2^n = D_3^n + D^n$, where

$$\begin{aligned} D_1^n &= -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t v_n, \quad D_2^n = \int_Q A(x,t,\nabla u_n)\nabla\left((1-\psi_\delta^-)\varphi\right), \quad D_3^n = \int_Q (1-\psi_\delta^-)\varphi d\widehat{\lambda_{n,0}} \\ D^n &= \int_Q (1-\psi_\delta^-)\varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1-\psi_\delta^-)\varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\ &= \int_Q (1-\psi_\delta^-)\varphi d(\rho_n - \eta_n). \end{aligned}$$

Clearly, $D_{i,m} - D_i^n = \omega(n,m)$ for $i = 1,2,3$. From Lemma (5.3) and (5.2)-(5.4), we obtain $D_{5,m} = \omega(n,m,\delta)$, and

$$\frac{1}{m}\int_{\{m\leqq v<2m\}} \psi_\delta^- \varphi A(x,t,\nabla u).\nabla v = \omega(n,m,\delta),$$

thus,

$$D_{4,m} = \frac{1}{m}\int_{\{m\leqq v<2m\}} \varphi A(x,t,\nabla u).\nabla v + \omega(n,m,\delta).$$

Since $\left|\int_Q (1-\psi_\delta^-)\varphi d\eta_n\right| \leqq \|\varphi\|_{L^\infty} \int_Q (1-\psi_\delta^-)d\eta_n$, it follows that $\int_Q (1-\psi_\delta^-)\varphi d\eta_n = \omega(n,m,\delta)$ from (5.4). And $\left|\int_Q \psi_\delta^- \varphi d\rho_n\right| \leqq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$, thus, from (5.2), $\int_Q (1-\psi_\delta^-)\varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta)$. Then $D^n = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta)$. Therefore by substraction, we get

$$\frac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+ + \omega(n,m,\delta),$$

hence

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+, \tag{5.33}$$

which proves (4.3) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\begin{aligned}
&\lim\nolimits_{m\to\infty}\tfrac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi A(x,t,\nabla u).\nabla v \\
&= \lim\nolimits_{m\to\infty}\tfrac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi\psi_\delta^+ A(x,t,\nabla u)\nabla v + \lim\nolimits_{m\to\infty}\tfrac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi(1-\psi_\delta^+)A(x,t,\nabla u).\nabla v \\
&= \textstyle\int_Q \varphi\psi_\delta^+ d\mu_s^+ + \lim\nolimits_{m\to\infty}\tfrac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi(1-\psi_\delta^+)A(x,t,\nabla u).\nabla v = \int_Q \varphi d\mu_s^+ + D,
\end{aligned}$$

where,

$$D = \int_Q \varphi(1-\psi_\delta^+)d\mu_s^+ + \lim_{n\to\infty}\frac{1}{m}\int\limits_{\{m\leqq v<2m\}} \varphi(1-\psi_\delta^+)A(x,t,\nabla u).\nabla v = \omega(\delta).$$

Therefore, (5.33) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (4.3) by density, and similarly, (4.4). This completes the proof of Theorem 2.1. ■

As a consequence of Theorem 2.1, we get the following:

Corollary 5.7 *Let $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q)$. Then there exists a R-solution u to the problem 1.1 with data (μ, u_0). Furthermore, if $v_0 \in L^1(\Omega)$ and $\omega \in \mathcal{M}_b(Q)$ such that $u_0 \leqq v_0$ and $\mu \leqq \omega$, then one can find R-solution v to the problem 1.1 with data (ω, v_0) such that $u \leqq v$.*

In particular, if $a \equiv 0$ in (1.2), then u satisfies (4.21) and $\|v\|_{L^\infty((0,T);L^1(\Omega))} \leqq M$ with $M = ||u_0||_{1,\Omega} + |\mu|(Q)$.

6 Equations with perturbation terms

Let $A : Q\times\mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.2), (1.3) with $a \equiv 0$. Let $\mathcal{G} : \Omega\times(0,T)\times\mathbb{R} \mapsto \mathbb{R}$ be a Caratheodory function. If U is a function defined in Q we define the function $\mathcal{G}(U)$ in Q by

$$\mathcal{G}(U)(x,t) = \mathcal{G}(x,t,U(x,t)) \qquad \text{for } a.e.\ (x,t)\in Q.$$

We consider the problem (1.5):

$$\begin{cases} u_t - \operatorname{div}(A(x,t,\nabla u)) + \mathcal{G}(u) = \mu & \text{in } Q, \\ u = 0 \qquad \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 \qquad \text{in } \Omega, \end{cases}$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$. We say that u is a R-solution of problem (1.5) if $\mathcal{G}(u) \in L^1(Q)$ and u is a R-solution of (1.1) with data $(\mu - \mathcal{G}(u), u_0)$.

6.1 Subcritical type results

For proving Theorem 2.2, we begin by an integration Lemma:

Lemma 6.1 *Let G satisfying (2.3). If a measurable function V in Q satisfies*

$$\text{meas}\,\{|V| \geqq t\} \leqq Mt^{-p_c}, \qquad \forall t \geqq 1,$$

for some $M > 0$, then for any $L > 1$,

$$\int_{\{|V| \geqq L\}} G(|V|) \leqq p_c M \int_L^\infty G(s)\, s^{-1-p_c} ds. \tag{6.1}$$

Proof. Indeed, setting $G_L(s) = \chi_{[L,\infty)}(s) G(s)$, we have

$$\int_{\{|V| \geqq L\}} G(|V|) dxdt = \int_Q G_L(|V|) dxdt \leqq \int_0^\infty G_L(|V|^*(s)) ds$$

where $|V|^*$ is and the rearrangement of $|V|$, defined by

$$|V|^*(s) = \inf\{a > 0 : \text{meas}\,\{|V| > a\}) \leqq s\}, \qquad \forall s \geqq 0.$$

From the assumption, we get $|V|^*(s) \leqq \sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)$. Thus, for any $L > 1$,

$$\int_{\{|V| \geqq L\}} G(|V|) dxdt \leqq \int_0^\infty G_L\left(\sup\left((Ms^{-1})^{p_c^{-1}}, 1\right)\right) ds = p_c M \int_L^\infty G(s)\, s^{-1-p_c} ds,$$

which implies (6.1). ■

Proof of Theorem 2.2. Proof of (i) Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, with $\mu_0 \in \mathcal{M}_0(Q), \mu_s \in \mathcal{M}_s(Q)$, and $u_0 \in L^1(\Omega)$. Then μ_0^+, μ_0^- can be decomposed as $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$. Let $\mu_{n,s,i} \in C_c^\infty(Q)$, $\mu_{n,s,i} \geqq 0$, converging respectively to μ_s^+, μ_s^- in the narrow topology. By Lemma 3.1, we can find $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$ which strongly converge to f_i, g_i, h_i in $L^1(Q)$, $\left(L^{p'}(Q)\right)^N$ and

X respectively, $i = 1, 2$, such that $\mu_0^+ = (f_1, g_1, h_1)$, $\mu_0^- = (f_2, g_2, h_2)$, and $\mu_{n,0,i} = (f_{n,i}, g_{n,i}, h_{n,i})$, converging respectively to μ_0^+, μ_0^- in the narrow topology. Furthermore, if we set

$$\mu_n = \mu_{n,0,1} - \mu_{n,0,2} + \mu_{n,s,1} - \mu_{n,s,2},$$

then $|\mu_n|(Q) \leq |\mu|(Q)$. Consider a sequence $\{u_{0,n}\} \subset C_c^\infty(\Omega)$ which strongly converges to u_0 in $L^1(\Omega)$ and satisfies $||u_{0,n}||_{1,\Omega} \leqq ||u_0||_{1,\Omega}$.

Let u_n be a solution of

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,\nabla u_n)) + \mathcal{G}(u_n) = \mu_n & \text{in } Q, \\ u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} \quad \text{in } \Omega. \end{cases}$$

We can choose $\varphi = \varepsilon^{-1} T_\varepsilon(u_n)$ as test function of above problem. Then we find

$$\int_Q \left(\varepsilon^{-1}\overline{T_\varepsilon}(u_n)\right)_t + \int_Q \varepsilon^{-1} A(x,t,\nabla T_\varepsilon(u_n)).\nabla T_\varepsilon(u_n) + \int_Q \mathcal{G}(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) = \int_Q \varepsilon^{-1}T_\varepsilon(u_n)d\mu_n.$$

Since

$$\int_Q \left(\varepsilon^{-1}\overline{T_\varepsilon}(u_n)\right)_t = \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_n(T))dx - \int_\Omega \varepsilon^{-1}\overline{T_\varepsilon}(u_{0,n})dx \geqq -||u_{0,n}||_{L^1(\Omega)},$$

there holds

$$\int_Q \mathcal{G}(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) \leqq |\mu_n|(Q) + ||u_{0,n}||_{L^1(\Omega)} \leqq |\mu|(Q) + ||u_0||_{1,\Omega}.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_Q |\mathcal{G}(x,t,u_n)| \leqq |\mu|(Q) + ||u_0||_{1,\Omega}. \tag{6.2}$$

Next apply Proposition 4.8 and Remark 4.9 to u_n with initial data $u_{0,n}$ and measure data $\mu_n - \mathcal{G}(u_n) \in L^1(Q)$, we get

$$\text{meas}\,\{|u_n| \geqq s\} \leqq C(|\mu|(Q) + ||u_0||_{L^1(\Omega)})^{\frac{p+N}{N}} s^{-p_c}, \qquad \forall s > 0, \forall n \in \mathbb{N},$$

for some $C = C(N, p, c_1, c_2)$. Since $|\mathcal{G}(x,t,u_n)| \leqq G(|u_n|)$, we deduce from (6.1) that $\{|\mathcal{G}(u_n)|\}$ is equi-integrable. Then, thanks to Proposition 4.10, up to a subsequence, $\{u_n\}$ converges to some function u, *a.e.* in Q, and $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$. Therefore, by Theorem 2.1, u is a R-solution of (2.4).

Proof of (ii). Let $\{u_n\}_{n \geqq 1}$ be defined by induction as nonnegative R-solutions of

$$\begin{cases} (u_1)_t - \operatorname{div}(A(x,t,\nabla u_1)) = \mu & \text{in } Q, \\ u_1 = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_1(0) = u_0 \quad \text{in } \Omega, \end{cases} \qquad \begin{cases} (u_{n+1})_t - \operatorname{div}(A(x,t,\nabla u_{n+1})) = \mu - \lambda\mathcal{G}(u_n) & \text{in } Q, \\ u_{n+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{n+1}(0) = u_0 \quad \text{in } \Omega, \end{cases}$$

Thanks to Corollary 5.7 we can assume that $\{u_n\}$ is nondecreasing and satisfies for any $s>0$ and $n\in\mathbb{N}$

$$\text{meas}\,\{|u_n|\geqq s\}\leqq C_1K_ns^{-p_c},\tag{6.3}$$

where C_1 does not depend on s,n, and

$$K_1=(||u_0||_{1,\Omega}+|\mu|(Q))^{\frac{p+N}{N}},$$
$$K_{n+1}=(||u_0||_{1,\Omega}+|\mu|(Q)+\lambda||\mathcal{G}(u_n)||_{1,Q})^{\frac{p+N}{N}},$$

for any $n\geqq 1$.Take $\varepsilon=\lambda+|\mu|(Q)+||u_0||_{L^1(\Omega)}\leqq 1$. Denoting by C_i some constants independent on n,ε, there holds $K_1\leqq C_2\varepsilon$, and for $n\geqq 1$,

$$K_{n+1}\leqq C_3\varepsilon(||\mathcal{G}(u_n)||_{1,Q}^{1+\frac{p}{N}}+1).$$

From (6.1) and (6.3), we find

$$\|\mathcal{G}(u_n)\|_{L^1(Q)}\leqq|Q|\,G(2)+\int\limits_{\{u_n|\geqq 2\}|}G(|u_n|)dxdt\leqq|Q|\,G(2)+C_4K_n\int_2^\infty G\,(s)\,s^{-1-p_c}ds.$$

Thus, $K_{n+1}\leqq C_5\varepsilon(K_n^{1+\frac{p}{N}}+1)$. Therefore, if ε is small enough, $\{K_n\}$ is bounded. Then, again from (6.1) and the relation $|\mathcal{G}(x,t,u_n)|\leqq G(|u_n|)$ we verify that $\{\mathcal{G}(u_n)\}$ converges. Then by Theorem 2.1, up to a subsequence, $\{u_n\}$ converges to a R-solution u of (2.5). ■

6.2 General case with absorption terms

In the sequel we assume that $A:\Omega\times\mathbb{R}^N\longmapsto\mathbb{R}^N$ does not depend on t. We recall a result obtained in [53],[17] in the elliptic case:

Theorem 6.2 *Let Ω be a bounded domain of $\mathbb{R}^N$. Let $A:\Omega\times\mathbb{R}^N\longmapsto\mathbb{R}^N$ satisfying (1.6),(1.7). Then there exists a constant κ depending on N,p,c_3,c_4 such that, if $\omega\in\mathcal{M}_b(\Omega)$ and u is a R-solution of problem*

$$\begin{cases}-div(A(x,\nabla u))=\omega & in\ \Omega,\\ u=0 \quad on\ \partial\Omega,\end{cases}$$

there holds

$$-\kappa\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^-]\leqq u\leqq\kappa\mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega^+].\tag{6.4}$$

Next we give a general result in case of absorption terms:

Theorem 6.3 *Let $p<N$, $A:\Omega\times\mathbb{R}^N\longmapsto\mathbb{R}^N$ satisfying (1.6),(1.7), and $\mathcal{G}:Q\times\mathbb{R}\longmapsto\mathbb{R}$ be a Caratheodory function such that the map $s\mapsto\mathcal{G}(x,t,s)$ is nondecreasing and odd, for a.e. (x,t) in Q.*

Let $\mu_1, \mu_2 \in \mathcal{M}_b^+(Q)$ such that there exist $\omega_n \in \mathcal{M}_b^+(\Omega)$ and nondecreasing sequences $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ in $\mathcal{M}_b^+(Q)$ with compact support in Q, converging to μ_1, μ_2, respectively in the narrow topology, and

$$\mu_{1,n}, \mu_{2,n} \leqq \omega_n \otimes \chi_{(0,T)}, \qquad \mathcal{G}((n + \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega_n])) \in L^1(Q),$$

where the constant c is given at Theorem 6.2. Let $u_0 \in L^1(\Omega)$, and $\mu = \mu_1 - \mu_2$. Then there exists a R-solution u of problem (1.5).

Moreover if $u_0 \in L^\infty(\Omega)$, and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$, for some $\gamma \in \mathcal{M}_b^+(\Omega)$, then a.e. in Q,

$$|u(x,t)| \leqq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\gamma(x) + ||u_0||_{\infty,\Omega}. \tag{6.5}$$

For proving this result, we need two Lemmas:

Lemma 6.4 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 6.3 and $\mathcal{G} \in L^\infty(Q \times \mathbb{R})$. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, and $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in \mathcal{M}_b^+(Q)$ with compact support in Q, $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω such that $\lambda_i \leqq \gamma \otimes \chi_{(0,T)}$. Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in Q. Then, there exist R-solutions u, u_1, u_2, to problems*

$$u_t - div(A(x, \nabla u)) + \mathcal{G}(u) = \lambda_1 - \lambda_2 \quad in\ Q, \qquad u = 0 \quad on\ \partial\Omega \times (0,T), \qquad u(0) = u_{0,1} - u_{0,2}, \tag{6.6}$$

$$(u_i)_t - div(A(x, \nabla u_i)) + \mathcal{G}(u_i) = \lambda_i \quad in\ Q, \qquad u_i = 0 \quad on\ \partial\Omega \times (0,T), \qquad u_i(0) = u_{0,i}, \tag{6.7}$$

relative to decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that a.e. in Q,

$$-||u_{0,2}||_{\infty,\Omega} - \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\gamma(x) \leqq -u_2(x,t) \leqq u(x,t) \leqq u_1(x,t) \leqq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\gamma(x) + ||u_{0,1}||_{\infty,\Omega}, \tag{6.8}$$

and

$$\int_Q |\mathcal{G}(u)| \leqq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}\right), \quad and \quad \int_Q \mathcal{G}(u_i) \leqq \lambda_i(Q) + ||u_{0,i}||_{1,\Omega}, \quad i = 1, 2. \tag{6.9}$$

Furthermore, assume that $\mathcal{H}, \mathcal{K}$ have the same properties as $\mathcal{G}$, and $\mathcal{H}(x,t,s) \leqq \mathcal{G}(x,t,s) \leqq \mathcal{K}(x,t,s)$ for any $s \in (0, +\infty)$ and a.e. in Q. Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to $\mathcal{H}, \mathcal{K}$ with data λ_i, such that $u_i(\mathcal{H}) \geqq u_i \geqq u_i(\mathcal{K})$, $i = 1, 2$.

Assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leqq \lambda_i \leqq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leqq u_{0,i} \leqq u_{0,i,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega_i, u_{0,i,2}), (\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leqq u_i \leqq u_i(\theta_i, u_{0,i,1})$.

Proof. Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, and $\varphi_n = \varphi_{1,n}\varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} * \gamma$, and for $i = 1, 2$, $u_{0,i,n} = \varphi_{2,n} * u_{0,i}$,

$$\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \mathrm{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},$$

where $f_{i,n} = \varphi_n * f_i$, $g_{i,n} = \varphi_n * g_i$, $h_{i,n} = \varphi_n * h_i$, $\lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$, and

$$\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \mathrm{div}(g_n) + (h_n)_t + \lambda_{s,n},$$

where $f_n = f_{1,n} - f_{2,n}$, $g_n = g_{1,n} - g_{2,n}$, $h_n = h_{1,n} - h_{2,n}$, $\lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n}$. Then for n large enough, $\lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q)$, $\gamma_n \in C_c^\infty(\Omega)$. Thus there exist unique solutions $u_n, u_{i,n}, v_{i,n}$, $i = 1, 2$, of problems

$$(u_n)_t - \operatorname{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) = \lambda_{1,n} - \lambda_{2,n} \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial\Omega\times(0,T), \quad u_n(0) = u_{0,1,n} - u_{0,2,n} \quad \text{in } \Omega,$$

$$(u_{i,n})_t - \operatorname{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \lambda_{i,n} \quad \text{in } Q, \qquad u_{i,n} = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad u_{i,n}(0) = u_{0,i,n} \quad \text{in } \Omega,$$

$$-\operatorname{div}(A(x, \nabla w_n)) = \gamma_n \quad \text{in } \Omega, \qquad w_n = 0 \quad \text{on } \partial\Omega,$$

such that

$$-||u_{0,2}||_{\infty,\Omega} - w_n(x) \leqq -u_{2,n}(x,t) \leqq u_n(x,t) \leqq u_{1,n}(x,t) \leqq w_n(x) + ||u_{0,1}||_{\infty,\Omega}, \quad a.e. \text{ in } Q.$$

Moreover, as in the Proof of Theorem 2.2, (i), there holds

$$\int_Q |\mathcal{G}(u_n)| \leqq \sum_{i=1,2} \left(\lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}\right), \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leqq \lambda_i(Q) + ||u_{0,i,n}||_{1,\Omega}, \quad i = 1, 2.$$

By Proposition 4.10, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converge to some (u, u_1, u_2), *a.e.* in Q. Since $\mathcal{G}$ is bounded, in particular, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ and $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, (6.9) is satisfied. Morover $\{\lambda_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\}$ and $\{\lambda_n - \mathcal{G}(u_n), f_n - \mathcal{G}(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\}$ are approximations of $(\lambda_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \lambda_{i,s}, u_{0,i})$ and $(\lambda - \mathcal{G}(u), f - \mathcal{G}(u), g, h, \lambda_s, u_{0,1} - u_{0,2})$, in the sense of Theorem 2.1. Thus, we can find (different) subsequences converging *a.e.* to u, u_1, u_2, R-solutions of (6.6) and (6.7). Furthermore, from [47, Corollary 3.4], up to a subsequence, $\{w_n\}$ converges *a.e.* in Q to a R-solution

$$-\operatorname{div}(A(x, \nabla w)) = \gamma \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega,$$

such that $w \leqq c\mathbf{W}_{1,p}^{2\text{diam}\Omega}\gamma$ *a.e.* in Ω. Hence, we get the inequality (6.8). The other conclusions follow in the same way. ■

Lemma 6.5 *Let $\mathcal{G}$ satisfy the assumptions of Theorem 6.3. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, $\lambda_i \in \mathcal{M}_b^+(Q)$ with compact support in Q, and $\gamma \in \mathcal{M}_b^+(\Omega)$ with compact support in Ω, such that*

$$\lambda_i \leqq \gamma \otimes \chi_{(0,T)}, \qquad \mathcal{G}((||u_{0,i}||_{\infty,\Omega} + \kappa\mathbf{W}_{1,p}^{2\text{diam}(\Omega)}\gamma)) \in L^1(Q). \tag{6.10}$$

Then, there exist R-solutions u, u_1, u_2 of the problems (6.6) and (6.7), respectively relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, satifying (6.8) and (6.9).

Moreover, assume that ω_i, θ_i have the same properties as λ_i and $\omega_i \leqq \lambda_i \leqq \theta_i$, $u_{0,i,1}, u_{0,i,2} \in L^{\infty+}(\Omega)$, $u_{0,i,2} \leqq u_{0,i} \leqq u_{0,i,1}$. Then, one can find solutions $u_i(\omega_i, u_{0,i,2})$, $u_i(\theta_i, u_{0,i,1})$, corresponding with $(\omega_i, u_{0,i,2})$, $(\theta_i, u_{0,i,1})$, such that $u_i(\omega_i, u_{0,i,2}) \leqq u_i \leqq u_i(\theta_i, u_{0,i,1})$.

Proof. From Lemma 6.4 there exist R-solutions u_n, $u_{i,n}$ to problems

$$(u_n)_t - \operatorname{div}(A(x,\nabla u_n)) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 \quad \text{in } Q, \qquad u_n = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad u_n(0) = u_{0,1} - u_{0,2}$$

$$(u_{i,n})_t - \operatorname{div}(A(x,\nabla u_{i,n})) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i \quad \text{in } Q, \qquad u_{i,n} = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad u_{i,n}(0) = u_{0,i},$$

relative to the decompositions $(f_1 - f_2 - T_n(\mathcal{G}(u_n), g_1 - g_2, h_1 - h_2)$, $(f_i - T_n(\mathcal{G}(u_{i,n}), g_i, h_i)$; and they satisfy

$$\begin{aligned} -||u_{0,2}||_{\infty,\Omega} - \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\gamma(x) \leqq -u_{2,n}(x,t) \leqq u_n(x,t) \\ \leqq u_{1,n}(x,t) \leqq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\gamma(x) + ||u_{0,1}||_{\infty,\Omega}, \end{aligned} \tag{6.11}$$

$$\int_Q |T_n(\mathcal{G}(u_n))| \leqq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i}||_{1,\Omega}), \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) \leqq \lambda_i(Q) + ||u_{0,i}||_{1,\Omega}.$$

As in Lemma 6.4, up to a common subsequence, $\{u_n, u_{1,n}, u_{2,n}\}$ converges *a.e.* in Q to $\{u, u_1, u_2\}$ for which (6.8) is satisfied *a.e.* in Q. From (6.10), (6.11) and the dominated convergence Theorem, we deduce that $\{T_n(\mathcal{G}(u_n))\}$ converges to $\mathcal{G}(u)$ and $\{T_n(\mathcal{G}(u_{i,n}))\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$. Thus, from Theorem 2.1, u and u_i are respective R-solutions of (6.6) and (6.7) relative to the decompositions $(f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2)$, $(f_i - \mathcal{G}(u_i), g_i, h_i)$, and (6.8) and (6.9 hold. The last statement follows from the same assertion in Lemma 6.4. ■

Proof of Theorem 6.3. By Proposition 3.2, for $i = 1,2$, there exist $f_{i,n}, f_i \in L^1(Q)$, $g_{i,n}, g_i \in (L^{p'}(Q))^N$ and $h_{i,n}, h_i \in X$, $\mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}_s^+(Q)$ such that

$$\mu_i = f_i - \operatorname{div} g_i + (h_i)_t + \mu_{i,s}, \qquad \mu_{i,n} = f_{i,n} - \operatorname{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},$$

and $\{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\}$ strongly converge to f_i, g_i, h_i in $L^1(Q)$, $(L^{p'}(Q))^N$ and X respectively, and $\{\mu_{i,n}\}, \{\mu_{i,n,s}\}$ converge to $\mu_i, \mu_{i,s}$ (strongly) in $\mathcal{M}_b(Q)$, and

$$||f_{i,n}||_{1,Q} + ||g_{i,n}||_{p',Q} + ||h_{i,n}||_X + \mu_{i,n,s}(\Omega) \leqq 2\mu(Q).$$

By Lemma 6.5, there exist R-solutions u_n, $u_{i,n}$ to problems

$$(u_n)_t - \operatorname{div}(A(x,\nabla u_n)) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} \quad \text{in } Q, \qquad u_n = 0 \quad \text{on } \partial\Omega\times(0,T), \quad u_n(0) = T_n(u_0) \tag{6.12}$$

$$(u_{i,n})_t - \operatorname{div}(A(x,\nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \mu_{i,n} \quad \text{in } \Omega, \qquad u_i = 0 \quad \text{on } \partial\Omega\times(0,T), \quad u_{i,n}(0) = T_n(u_0^{\pm}), \tag{6.13}$$

for $i = 1,2$, relative to the decompositions $(f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})$, such that $\{u_{i,n}\}$ is nonnegative and nondecreasing, and $-u_{2,n} \leqq u_n \leqq u_{1,n}$; and

$$\int_Q |\mathcal{G}(u_n)| \leqq \mu_1(Q) + \mu_2(Q) + ||u_0||_{1,\Omega}, \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leqq \mu_i(Q) + ||u_0||_{1,\Omega}, \quad i = 1,2. \tag{6.14}$$

As in the proof of Lemma 6.5, up to a common subsequence $\{u_n, u_{1,n}, u_{2,n}\}$ converge *a.e.* in Q to $\{u, u_1, u_2\}$. Since $\{\mathcal{G}(u_{i,n})\}$ is nondecreasing, and nonnegative, from the monotone convergence

Theorem and (6.14), we obtain that $\{\mathcal{G}(u_{i,n})\}$ converges to $\mathcal{G}(u_i)$ in $L^1(Q)$, $i=1,2$. Finally, $\{\mathcal{G}(u_n)\}$ converges to $\mathcal{G}(u)$ in $L^1(Q)$, since $|\mathcal{G}(u_n)| \leqq \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n})$. Thus, we can see that

$$\left\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_0^+) - T_n(u_0^-)\right\}$$

is an approximation of $(\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)$, in the sense of Theorem 2.1; and

$$\left\{\mu_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \mu_{i,s,n}, T_n(u_0^{\pm})\right\}$$

is an approximation of $(\mu_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \mu_{i,s}, u_0^{\pm})$. Therefore, u is a R-solution of (1.5), and (6.5) holds if $u_0 \in L^\infty(\Omega)$ and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$ and some $\gamma \in \mathcal{M}_b^+(\Omega)$. ■

As a consequence we prove Theorem 2.3. We use the following result of [17]:

Proposition 6.6 (see [17]) *Let $q > p-1$, $\alpha \in \left(0, \frac{N(q+1-p)}{pq}\right)$, $r > 0$ and $\nu \in \mathcal{M}_b^+(\Omega)$. If ν does not charge the sets of $C_{\alpha p, \frac{q}{q+1-p}}$-capacity zero, there exists a nondecreasing sequence $\{\nu_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ν strongly in $\mathcal{M}_b(\Omega)$ and such that $\mathbf{W}_{\alpha,p}^r[\nu_n] \in L^q(\mathbb{R}^N)$, for any $n \in \mathbb{N}$.*

Proof of Theorem 2.3. Let $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \leqq \omega \otimes F$, where $F \in L^1((0,T))$ and ω does not charge the sets of $C_{p, \frac{q}{q+1-p}}$-capacity zero. From Proposition 6.6, there exists a nondecreasing sequence $\{\omega_n\} \subset \mathcal{M}_b^+(\Omega)$ with compact support in Ω which converges to ω, strongly in $\mathcal{M}_b(\Omega)$, such that $\mathbf{W}_{1,p}^{2diam\Omega}[\omega_n] \in L^q(\mathbb{R}^N)$. We can write

$$f + \mu = \mu_1 - \mu_2, \qquad \mu_1 = f^+ + \mu^+, \qquad \mu_2 = f^- + \mu^-, \tag{6.15}$$

and $\mu^+, \mu^- \leqq \omega \otimes F$. We set

$$Q_n = \{(x,t) \in \Omega \times (\frac{1}{n}, T - \frac{1}{n}) : d(x, \partial\Omega) > \frac{1}{n}\}, \qquad F_n = T_n(\chi_{(\frac{1}{n}T - \frac{1}{n})}F), \tag{6.16}$$

$$\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \qquad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}. \tag{6.17}$$

Then $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ are nondecreasing sequences with compact support in Q, and $\mu_{1,n}, \mu_{2,n} \leqq \tilde{\omega}_n \otimes \chi_{(0,T)}$, with $\tilde{\omega}_n = n(\chi_\Omega + \omega_n)$ and $(n + \kappa \mathbf{W}_{1,p}^{2diam\Omega}[\omega_n])^q \in L^1(Q)$. Besides, $\omega_n \otimes F_n$ converges to $\omega \otimes F$ strongly in $\mathcal{M}_b(Q)$: indeed we easily check that

$$||\omega_n \otimes F_n - \omega \otimes F||_{\mathcal{M}_b(Q)} \leqq ||F_n||_{L^1((0,T))} ||\omega_n - \omega||_{\mathcal{M}_b(\Omega)} + ||\omega||_{\mathcal{M}_b(\Omega)} ||F_n - F||_{L^1((0,T))}$$

Observe that for any measures $\nu, \theta, \eta \in \mathcal{M}_b(Q)$, there holds

$$|\inf\{\nu, \theta\} - \inf\{\nu, \eta\}| \leqq |\theta - \eta|,$$

hence $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ converge to μ_1, μ_2 respectively in $\mathcal{M}_b(Q)$. Therefore, the result follows from Theorem 6.3. ■

Remark 6.7 *Our result improves the existence results of [50], where* $\mu \in \mathcal{M}_0(Q)$. *Indeed, let* $p_e = N(p-1)/(N-p)$ *be the critical exponent for the elliptic problem*

$$-\Delta_p w + |w|^{q-1} w = \omega \quad in\ \Omega, \qquad w = 0 \quad on\ \partial\Omega.$$

Notice that $p_c < p_e$, *since* $p > p_1$. *If* $q \geqq p_e$, *there exist measures* $\omega \in \mathcal{M}_b^+(\Omega)$ *which do not charge the sets of* $C_{p,\frac{q}{q+1-p}}$*-capacity zero, such that* $\omega \notin \mathcal{M}_{0,e}(\Omega)$. *Then for any* $F \in L^1((0,T))$, $F \geqq 0, F \not\equiv 0$, *we have* $\omega \otimes F \notin \mathcal{M}_0(Q)$.

Remark 6.8 *Let* $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ *satisfying (1.6),(1.7). Let* $\mathcal{G} : Q \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function such that the map* $s \mapsto \mathcal{G}(x,t,s)$ *is nondecreasing and odd, for a.e.* (x,t) *in* Q. *Assume that* $\omega \in \mathcal{M}_{0,e}(\Omega)$. *Thus, we have* $\omega(\{x : W_{1,p}^{2diam(\Omega)}[\omega](x) = \infty\}) = 0$. *As in the proof of Theorem 2.3 with* $\omega_n = \chi_{W_{1,p}^{2diam\Omega}[\omega] \leqq n}\omega$, *we get that (1.5) has a R-solution.*

Remark 6.9 *As in [17], from Theorem 6.3, we can extend Theorem 2.3 given for* $\mathcal{G}(u) = |u|^{q-1} u$, *to the case of a function* $\mathcal{G}(x,t,.)$, *odd for a.e.* $(x,t) \in Q$, *such that*

$$|\mathcal{G}(x,t,u)| \leqq G(|u|), \qquad \int_1^\infty G(s)s^{-q-1}ds < \infty,$$

where G *is a nondecreasing continuous, under the condition that* ω *does not charge the sets of zero* $C_{p,\frac{q}{q-p+1},1}$*-capacity, where for any Borel set* $E \subset \mathbb{R}^N$,

$$C_{p,\frac{q}{q-p+1},1}(E) = \inf\{||\varphi||_{L^{\frac{q}{q-p+1},1}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1},1}(\mathbb{R}^N), \quad G_p * \varphi \geqq \chi_E\}$$

where $L^{\frac{q}{q-p+1},1}(\mathbb{R}^N)$ *is the Lorentz space of order* $(q/(q-p+1),1)$.

In case $\mathcal{G}$ is of exponential type, we introduce the notion of maximal fractional operator, defined for any $\eta \geqq 0$, $R > 0$, $x_0 \in \mathbb{R}^N$ by

$$\mathbf{M}_{p,R}^{\eta}[\omega](x_0) = \sup_{t\in(0,R)} \frac{\omega(B(x_0,t))}{t^{N-p}h_\eta(t)}, \qquad \text{where } h_\eta(t) = \inf((-\ln t)^{-\eta}, (\ln 2)^{-\eta})).$$

We obtain the following:

Theorem 6.10 *Let* $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ *satisfying (1.6),(1.7). Let* $p < N$ *and* $\tau > 0, \beta > 1, \mu \in \mathcal{M}_b(Q)$ *and* $u_0 \in L^1(\Omega)$. *Assume that* $|\mu| \leqq \omega \otimes F$, *with* $\omega \in \mathcal{M}_b^+(\Omega)$, $F \in L^1((0,T))$ *be nonnegative. Assume that one of the following assumptions is satisfied:*
(i) $||F||_{L^\infty((0,T))} \leqq 1$ *and for some* $M_0 = M_0(N,p,\beta,\tau,c_3,c_4,\mathrm{diam}\Omega)$,

$$||\mathbf{M}_{p,2\mathrm{diam}_\Omega}^{\frac{p-1}{\beta'}}[\omega]||_{L^\infty(\mathbb{R}^N)} < M_0, \tag{6.18}$$

(ii) there exists $\beta_0 > \beta$ *such that* $\mathbf{M}_{p,2diam\Omega}^{\frac{p-1}{\beta_0'}}[\omega] \in L^\infty(\mathbb{R}^N)$.

Then there exists a R-solution to the problem

$$\begin{cases} u_t - div(A(x,\nabla u)) + (e^{\tau|u|^\beta} - 1)\mathrm{sign}u = F + \mu & \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases}$$

For the proof we use the following result of [17]:

Proposition 6.11 (see [17], Theorem 2.4) *Suppose* $1 < p < N$. *Let* $\nu \in \mathcal{M}_b^+(\Omega)$, $\beta > 1$, *and* $\delta_0 = ((12\beta)^{-1})^\beta p \ln 2$. *There exists* $C = C(N,p,\beta,\mathrm{diam}\Omega)$ *such that, for any* $\delta \in (0,\delta_0)$,

$$\int_\Omega \exp\left(\delta \frac{(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\nu])^\beta}{||\mathbf{M}_{p,2\mathrm{diam}_\Omega}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}}}\right) \leqq \frac{C}{\delta_0 - \delta}.$$

Proof of Theorem 6.10. Let Q_n be defined at (6.16), and $\omega_n = \omega\chi_{\Omega_n}$, where $\Omega_n = \{x \in \Omega : d(x,\partial\Omega) > 1/n\}$. We still consider $\mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n}$ as in (6.15), (6.17).
Case 1: Assume that $||F||_{L^\infty((0,T))} \leqq 1$ and (6.18) holds. We have $\mu_{1,n}, \mu_{2,n} \leqq n\chi_\Omega + \omega$. For any $\varepsilon > 0$, there exists $c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, \mathrm{diam}\Omega) > 0$ such that

$$(n + \kappa\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[n\chi_\Omega + \omega])^\beta \leqq c_\varepsilon n^{\frac{\beta p}{p-1}} + (1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^\beta$$

a.e. in Ω. Thus,

$$\exp\left(\tau(n + \kappa\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[n\chi_\Omega + \omega])^\beta\right) \leqq \exp\left(\tau c_\varepsilon n^{\frac{\beta p}{p-1}}\right)\exp\left(\tau(1+\varepsilon)\kappa^\beta(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^\beta\right)$$

If (6.18) holds with $M_0 = \left(\delta_0/\tau\kappa^\beta\right)^{(p-1)/\beta}$ then we can chose ε such that

$$\tau(1+\varepsilon)\kappa^\beta||\mathbf{M}_{p,2\mathrm{diam}_\Omega}^{\frac{p-1}{\beta'}}[\nu]||_{L^\infty(\mathbb{R}^N)}^{\frac{\beta}{p-1}} < \delta_0.$$

From Proposition 6.11, we get $\exp(\tau(1+\varepsilon)\kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^\beta) \in L^1(\Omega)$, which implies $\exp(\tau(n + \kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[n\chi_\Omega + \omega])^\beta) \in L^1(\Omega)$ for all n. We conclude from Theorem 6.3.
Case 2: Assume that there exists $\varepsilon > 0$ such that $\mathbf{M}_{p,2diam\Omega}^{(p-1)/(\beta+\varepsilon)'}[\omega] \in L^\infty(\mathbb{R}^N)$. Now we use the inequality $\mu_{1,n}, \mu_{2,n} \leqq n(\chi_\Omega + \omega)$. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $c_{\varepsilon,n} > 0$ such that

$$(n + \kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta \leqq c_{\varepsilon,n} + \varepsilon(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^{\beta_0}$$

Thus, from Proposition 6.11 we get $\exp(\tau(n + \kappa^\beta\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[n(\chi_\Omega + \omega)])^\beta) \in L^1(\Omega)$ for all n. We conclude from Theorem 6.3. ∎

6.3 Equations with source term

As a consequence of Theorem 6.3, we get a first result for problem (1.1):

Corollary 6.12 *Let $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ satisfying (1.6)(1.7). Let $u_0 \in L^\infty(\Omega)$, and $\mu \in \mathcal{M}_b(Q)$ such that $|\mu| \leqq \omega \otimes \chi_{(0,T)}$ for some $\omega \in \mathcal{M}_b^+(\Omega)$. Then there exist a R-solution u of (1.1), such that*

$$|u(x,t)| \leqq \kappa \mathbf{W}_{1,p}^{2\mathrm{diam}(\Omega)}[\omega](x) + ||u_0||_{\infty,\Omega}, \qquad \text{for a.e. } (x,t) \in Q, \tag{6.19}$$

where κ is defined at Theorem 6.2.

Proof. Let $\{\phi_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Since $\{\phi_n\mu^+\}, \{\phi_n\mu^-\}$ are nondecreasing sequences, the result follows from Theorem 6.3. ■

Our proof of Theorem 2.4 is based on a property of Wölf potentials:

Theorem 6.13 (see [53]) *Let $q > p-1$, $0 < p < N$, $\omega \in \mathcal{M}_b^+(\Omega)$. If for some $\lambda > 0$,*

$$\omega(E) \leqq \lambda C_{p,\frac{q}{p-q+1}}(E) \qquad \text{for any compact set } E \subset \mathbb{R}^N, \tag{6.20}$$

then $(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^q \in L^1(\Omega)$, and there exists $M = M(N,p,q,\mathrm{diam}(\Omega))$ such that, a.e. in Ω,

$$\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\left[\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega]\right]^q \leqq M\lambda^{\frac{q-p+1}{(p-1)^2}}\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega] < \infty. \tag{6.21}$$

We deduce the following:

Lemma 6.14 *Let $\omega \in \mathcal{M}_b^+(\Omega)$, and $b \geqq 0$ and $K > 0$. Suppose that $\{u_m\}_{m\geqq 1}$ is a sequence of nonnegative functions in Ω that satisfies*

$$u_1 \leqq K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega] + b, \qquad u_{m+1} \leqq K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[u_m^q + \omega] + b \qquad \forall m \geqq 1.$$

Assume that ω satisfies (6.20) for some $\lambda > 0$. Then there exist λ_0 and b_0, depending on N, p, q, K, and $\mathrm{diam}\Omega$, such that, if $\lambda \leqq \lambda_0$ and $b \leqq b_0$, then $\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\mu] \in L^q(\Omega)$ and for any $m \geqq 1$,

$$u_m \leqq 2\beta_p K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega] + 2b, \qquad \beta_p = \max(1, 3^{\frac{2-p}{p-1}}). \tag{6.22}$$

Proof. Clearly, (6.22) holds for $m = 1$. Now, assume that it holds at the order m. Then

$$u_m^q \leqq 2^{q-1}(2\beta_p)^q K^q(\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega])^q + 2^{q-1}(2b)^q$$

Using (6.21) we get

$$\begin{aligned}
u_{m+1} &\leq K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\left[2^{q-1}(2\beta_p)^q K^q(W_{1,p}^{2\mathrm{diam}\Omega}[\omega])^q + 2^{q-1}(2b)^q + \omega\right] + b \\
&\leq \beta_p K\left(A_1\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\left[(W_{1,p}^{2\mathrm{diam}\Omega}[\omega])^q\right] + \mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\left[(2b)^q\right] + W_{1,p}^{2\mathrm{diam}\Omega}[\omega]\right) + b \\
&\leq \beta_p K(A_1 M\lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega] + \beta_p K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}\left[(2b)^q\right] + b \\
&= \beta_p K(A_1 M\lambda^{\frac{q-p+1}{(p-1)^2}} + 1)\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega] + A_2 b^{\frac{q}{p-1}} + b,
\end{aligned}$$

where M is as in (6.21) and $A_1 = \left(2^{q-1}(2\beta_p)^q K^q\right)^{1/(p-1)}$, $A_2 = \beta_p K 2^{q/(p-1)} |B_1|^{1/(p-1)} (p')^{-1} (2\text{diam}\Omega)^{p'}$. Thus, (6.22) holds for $m = n+1$ if we prove that

$$A_1 M \lambda^{\frac{q-p+1}{(p-1)^2}} \leq 1 \text{ and } A_2 b^{\frac{q}{p-1}} \leq b,$$

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}} \text{ and } b \leq A_2^{-\frac{p-1}{q-p+1}}.$$

Therefore, we obtain the result with $\lambda_0 = (A_1 M)^{-(p-1)^2/(q-p+1)}$ and $b_0 = A_2^{-(p-1)/(q-p+1)}$. ■

Proof of Theorem 2.4. From Corollary 5.7 and 6.12, we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m \geqq 1}$ defined in the following way: u_1 is a R-solution of (1.1), and u_{m+1} is a nonnegative R-solution of

$$\begin{cases} (u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = u_m^q + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{m+1}(0) = u_0 \quad \text{in } \Omega. \end{cases}$$

Setting $\overline{u}_m = \sup_{t \in (0,T)} u_m(t)$ for all $m \geqq 1$, there holds

$$\overline{u}_1 \leqq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + ||u_0||_{\infty,\Omega}, \qquad \overline{u}_{m+1} \leqq \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\overline{u}_m^q + \omega] + ||u_0||_{\infty,\Omega} \qquad \forall m \geqq 1.$$

From Lemma 6.14, we can find $\lambda_0 = \lambda_0(N, p, q, diam\Omega)$ and $b_0 = b_0(N, p, q, \text{diam}\Omega)$ such that if (2.7) is satisfied with λ_0 and b_0, then

$$u_m \leqq \overline{u}_m \leqq 2\beta_p \kappa \mathbf{W}_{1,p}^{2\text{diam}\Omega}[\omega] + 2||u_0||_{\infty,\Omega} \qquad \forall m \geqq 1. \tag{6.23}$$

Thus $\{u_m\}$ converges *a.e.* in Q and in $L^1(Q)$ to some function u, for which (2.9) is satisfied in Ω with $c = 2\beta_p\kappa$. Finally, one can apply Theorem 2.1 to the sequence of measures $\{u_m^q + \mu\}$, and obtain that u is a R-solution of (2.8). ■

Next we consider the exponential case.

Theorem 6.15 *Let $A : \Omega \times \mathbb{R}^N \longmapsto \mathbb{R}^N$ satisfying (1.6),(1.7). Let $\tau > 0, l \in \mathbb{N}$ and $\beta \geqq 1$ such that $l\beta > p-1$. Set*

$$\mathcal{E}(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \qquad \forall s \in \mathbb{R}. \tag{6.24}$$

Let $\mu \in \mathcal{M}_b^+(Q)$, $\omega \in \mathcal{M}_b^+(\Omega)$ such that $\mu \leqq \chi_{(0,T)} \otimes \omega$. Then, there exist b_0 and M_0 depending on N, p, β, τ, l and diamΩ, *such that if*

$$||\mathbf{M}_{p,2\text{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\omega]||_{L^\infty(\mathbb{R}^N)} \leqq M_0, \qquad ||u_0||_{\infty,\Omega} \leqq b_0,$$

the problem

$$\begin{cases} u_t - div(A(x, \nabla u)) = \mathcal{E}(\tau u^\beta) + \mu & in\ Q, \\ u = 0 \quad on\ \partial\Omega \times (0,T), \\ u(0) = u_0 \quad in\ \Omega \end{cases} \tag{6.25}$$

admits nonnegative R- solution u, which satisfies, a.e. in Q, for some c, depending on N, p, c_3, c_4

$$u(x,t) \leqq c\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\omega](x) + 2b_0. \tag{6.26}$$

For the proof we first recall an approximation property, which is a consequence of [47, Theorem 2.5]:

Theorem 6.16 *Let $\tau > 0$, $b \geqq 0$, $K > 0$, $l \in \mathbb{N}$ and $\beta \geqq 1$ such that $l\beta > p - 1$. Let $\mathcal{E}$ be defined by (6.24). Let $\{v_m\}$ be a sequence of nonnegative functions in Ω such that, for some $K > 0$,*

$$v_1 \leqq K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\mu] + b, \qquad v_{m+1} \leqq K\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\mathcal{E}(\tau u_m^\beta) + \mu] + b, \quad \forall m \geqq 1.$$

Then, there exist b_0 and M_0, depending on N, p, β, τ, l, K and $\mathrm{diam}\Omega$ such that if $b \leqq b_0$ and

$$||\mathbf{M}_{p,2\mathrm{diam}\Omega}^{\frac{(p-1)(\beta-1)}{\beta}}[\mu]||_{\infty,\mathbb{R}^N} \leqq M_0, \tag{6.27}$$

then, setting $c_p = 2\max(1,2^{\frac{2-p}{p-1}})$,

$$\exp(\tau(Kc_p\mathbf{W}_{1,p}^{2\mathrm{diam}\Omega}[\mu] + 2b_0)^\beta) \in L^1(\Omega),$$

$$v_m \leqq Kc_pW_{1,p}^{2\mathrm{diam}\Omega}[\mu] + 2b_0, \quad \forall m \geqq 1. \tag{6.28}$$

Proof of Theorem 6.15. From Corollary 5.7 and 6.12 we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m\geqq 1}$ defined in the following way: u_1 is a R-solution of problem (1.1), and by induction, u_{m+1} is a R-solution of

$$\begin{cases} (u_{m+1})_t - \mathrm{div}(A(x, \nabla u_{m+1})) = \mathcal{E}(\tau u_m^\beta) + \mu & \text{in } Q, \\ u_{m+1} = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u_{m+1}(0) = u_0 \quad \text{in } \Omega. \end{cases}$$

And, setting $\overline{u}_m = \sup_{t\in(0,T)} u_m(t)$, there holds

$$\overline{u}_1 \leqq \kappa W_{1,p}^{2\mathrm{diam}\Omega}[\omega] + ||u_0||_{\infty,\Omega}, \qquad \overline{u}_{m+1} \leqq \kappa W_{1,p}^{2\mathrm{diam}\Omega}[\mathcal{E}(\tau\overline{u}_m^\beta) + \omega] + ||u_0||_{\infty,\Omega}, \qquad \forall m \geqq 1.$$

Thus, from Theorem 6.16, there exist $b_0 \in (0,1]$ and $M_0 > 0$ depending on N, p, β, τ, l and $\mathrm{diam}\Omega$ such that, if (6.27) holds, then (6.28) is satisfied with $v_m = \overline{u}_m$. As a consequence, u_m is well defined. Thus, $\{u_m\}$ converges *a.e.* in Q to some function u, for which (6.26) is satisfied in Ω. Furthermore, $\left\{\mathcal{E}(\tau u_m^\beta)\right\}$ converges to $\mathcal{E}(\tau u^\beta)$ in $L^1(Q)$. Finally, one can apply Theorem 2.1 to the sequence of measures $\left\{\mathcal{E}(\tau u_m^\beta) + \mu\right\}$, and obtain that u is a R-solution of (6.25). ■

7 Appendix

Proof of Lemma 4.7. Let $\mathcal{J}$ be defined by (4.11). Let $\zeta \in C_c^1([0,T))$ with values in $[0,1]$, such that $\zeta_t \leqq 0$, and $\varphi = \zeta\xi[j(S(v))]_l$. Clearly, $\varphi \in X \cap L^\infty(Q)$; we choose the pair of functions (φ, S) as test function in (4.2). Thanks to convergence properties of Steklov time-averages, we easily will obtain (4.15) if we prove that

$$\underline{\lim}_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[j(S(v))]_l)_t S(v)) \geqq -\int_Q \xi_t J(S(v)).$$

We can write $-\int_Q (\zeta\xi[j(S(v))]_l)_t S(v) = F + G$, with

$$F = -\int_Q (\zeta\xi)_t[j(S(v))]_l S(v), \qquad G = -\int_Q \zeta\xi S(v)\frac{1}{l}\left(j(S(v))(x,t+l) - j(S(v))(x,t)\right).$$

Using (4.12) and integrating by parts we have

$$\begin{aligned}
G &\geqq -\int_Q \zeta\xi\frac{1}{l}\left(\mathcal{J}(S(v))(x,t+l) - \mathcal{J}(S(v))(x,t)\right) \\
&= -\int_Q \zeta\xi\frac{\partial}{\partial t}\left([\mathcal{J}(S(v))]_l\right) = \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l + \int_\Omega \zeta(0)\xi(0)[\mathcal{J}(S(v))]_l(0) \\
&\geqq \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l,
\end{aligned}$$

since $\mathcal{J}(S(v)) \geqq 0$. Hence,

$$-\int_Q (\zeta\xi[j(S(v))]_l)_t S(v) \geqq \int_Q (\zeta\xi)_t[\mathcal{J}(S(v))]_l + F = \int_Q (\zeta\xi)_t\left([\mathcal{J}(S(v))]_l - [J(S(v))]_l S(v)\right)$$

Otherwise, $\mathcal{J}(S(v))$ and $J(S(v) \in C([0,T]\,; L^1(\Omega))$, thus $\{(\zeta\xi)_t\left([\mathcal{J}(S(u))]_l - [J(S(u))]_l S(u)\right)\}$ converges to $-(\zeta\xi)_t J(S(u))$ in $L^1(Q)$ as $l \to 0$. Therefore,

$$\begin{aligned}
\underline{\lim}_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[J(S(v))]_l)_t S(v)) &\geqq \underline{\lim}_{\zeta\to 1}\left(-\int_Q (\zeta\xi)_t J(S(v))\right) \\
&\geqq \underline{\lim}_{\zeta\to 1}\left(-\int_Q \zeta\xi_t J(S(v))\right) = -\int_Q \xi_t J(S(v)),
\end{aligned}$$

which achieves the proof. ■

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